

SECOND MOMENTS IN THE GENERALIZED GAUSS CIRCLE PROBLEM

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ABSTRACT. The generalized Gauss circle problem concerns the lattice point discrepancy of large spheres. In this paper, we study the Dirichlet series associated to $P_k(n)^2$, where $P_k(n)$ is the discrepancy between the volume of the k -dimensional sphere of radius \sqrt{n} and the number of integer lattice points contained in that sphere. Using this Dirichlet series, we prove asymptotics with improved power-saving error terms for smoothed sums, including $\sum P_k(n)^2 e^{-n/X}$ and the Laplace transform $\int_0^\infty P_k(t)^2 e^{-t/X} dt$, in dimensions $k \geq 3$.

By taking combinations of these smoothed sums, we show that the sharp sum $\sum_{n \leq X} P_k(n)^2$ has one main term and a power-savings error term for dimensions $k \geq 4$, and two main terms and a power-savings error term in dimension-three. We also prove similar results for the sharp integral $\int_0^X P_3(t)^2 dt$, producing the first power-savings error term in mean square estimates for the dimension 3 Gauss circle problem since Jarnik identified any separation between the main term and error in 1940.

1. INTRODUCTION

Let $r_k(m)$ denote the number of integer k -tuples (n_1, n_2, \dots, n_k) such that $n_1^2 + \dots + n_k^2 = m$, and let $S_k(n)$ denote the sum of $r_k(m)$ for $m \leq n$,

$$S_k(n) = \sum_{0 \leq m \leq n} r_k(m).$$

Geometrically, $S_k(n)$ counts the number of lattice points in \mathbb{Z}^k contained within $B_k(\sqrt{n})$, the k -dimensional sphere of radius \sqrt{n} . It is intuitively clear that $S_k(n) \sim \text{Vol}(B_k(\sqrt{n}))$ as $n \rightarrow \infty$.

To describe this asymptotic more precisely, set

$$S_k(n) = \text{Vol}(B_k(\sqrt{n})) + P_k(n). \quad (1.1)$$

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In the $k = 2$ case, estimation of $P_k(n)$ is the famous *Gauss circle problem*. Here, Gauss established $P_2(n) = O(\sqrt{n})$ by relating $P_2(n)$ to the area of a narrow annulus enclosing the boundary of $B_2(\sqrt{n})$ [IKKN06].

For general $k \geq 2$, the pursuit of a minimal exponent α_k for which $P_k(n) = O(n^{\alpha_k + \epsilon})$ for any $\epsilon > 0$ is now known as the *generalized Gauss circle problem*. Gauss' geometric argument readily generalizes to show that $\alpha_k \leq (k-1)/2$, but Ω -type results (see [IKKN06] for a survey) support the conjecture that

$$\alpha_k = \begin{cases} \frac{1}{4}, & k = 2 \\ \frac{k}{2} - 1, & k > 2 \end{cases} \quad (1.2)$$

are the true sizes. For $k \geq 4$, this conjecture is known to be true, and for $k \geq 5$ the order of growth of $P_k(n)$ is known (up to constants), as described in [Kr00].

Far less is known in the case $k \leq 3$. In the case $k = 2$, the first improvement on Gauss' result is due to Sierpiński [Sie06], who established $P_2(n) = O(n^{\frac{1}{3}})$ using Poisson summation and the theory of exponential sums. Incremental progress has led to Huxley's *discrete Hardy-Littlewood method* [Hux03] and the result

$$P_2(n) = O\left(n^{131/416}(\log n)^{18637/8320}\right).$$

Notable progress for $k = 3$ includes Landau's result $P_3(n) = O(n^{3/4})$ [Lan19] and a long series of results due to Vinogradov culminating in $P_3(n) = O(n^{2/3}(\log n)^6)$ [Vin63]. The current best result is due to Heath-Brown [HB99], who obtained

$$P_3(n) = O\left(n^{\frac{21}{32} + \epsilon}\right).$$

Some of the best evidence for the conjectured exponents (1.2) in the generalized Gauss circle problem is given by *mean square* results describing

$$\int_0^X (P_k(x))^2 dx.$$

In the case $k = 2$ the best known result is due to Nowak [Now04], who proved

$$\int_0^X (P_2(x))^2 dx = \frac{1}{3\pi^2} \sum_{n=1}^{\infty} \frac{r_2^2(n)}{n^{3/2}} X^{3/2} + O\left(X(\log X)^{3/2} \log \log X\right).$$

In the case $k = 3$, the main result of this form is an old result due to Jarník [Jar40], who established

$$\int_0^X (P_3(x))^2 dx = c_3 X^2 \log X + O\left(X^2 (\log X)^{1/2}\right) \quad (1.3)$$

for some $c_3 > 0$ using the Hardy-Littlewood method. This error was more recently improved to $O(X^2)$ by Lau [Lau99]. For $k \geq 4$, Jarník proved mean

square results with power-savings error terms of the form

$$\int_0^X (P_k(x))^2 dx = c_k X^{k-1} + O(g(X)), \quad (1.4)$$

with

$$g(X) = \begin{cases} X^{\frac{5}{2}} \log X & \text{if } k = 4, \\ X^3 \log^2 X & \text{if } k = 5, \\ X^{k-2} & \text{if } k > 5. \end{cases}$$

The relatively large error term in dimension three suggests that this case is the most mysterious and least understood. For $k > 5$, these results are optimal, while for $k \leq 5$ these bounds may be improved and it may be possible to extract additional lower order terms. More detail on progress towards the generalized Gauss circle problem and its many cousins can be found in the excellent survey [IKKN06].

In this paper, we consider mean square estimates for the generalized Gauss circle problem, focusing on the cases $k > 2$. Our first result is a mean square estimate with exponential smoothing.

Theorem 1.1. *For $k \geq 3$ and any $\epsilon > 0$,*

$$\begin{aligned} \sum_{n=1}^{\infty} P_k(n)^2 e^{-n/X} &= \delta_{[k=3]} C'_3 X^{k-1} (\log X + 1 - \gamma) + C_k \Gamma(k-1) X^{k-1} \\ &\quad + \delta_{[k=4]} C'_4 \Gamma(k - \tfrac{3}{2}) X^{k-\frac{3}{2}} + O_{\epsilon}(X^{k-2+\epsilon}), \end{aligned} \quad (1.5)$$

where C_k , C'_3 , and C'_4 are explicit constants, and

$$\delta_{[k=n]} = \begin{cases} 0 & \text{if } k \neq n, \\ 1 & \text{if } k = n \end{cases}$$

is a Kronecker delta indicator function.

Remark 1.2. The coefficients C'_3 , C'_4 , and C_k ($k \geq 4$) are given by

$$\begin{aligned} C'_3 &= \frac{\pi^2}{3\zeta^{(2)}(3)}, \quad C'_4 = \frac{16(9\sqrt{2} - 8)\zeta(\frac{1}{2})\zeta(\frac{3}{2})^2\zeta(\frac{5}{2})}{7\pi^2\zeta(3)}, \\ C_k &= \frac{k^2}{24} \text{Vol}(B_k(1))^2 + \frac{\pi^k \zeta(k-2)}{12 \Gamma(\frac{k}{2})^2 \zeta^{(2)}(k)} \left(1 + 2^{3-k}\right). \end{aligned}$$

The size of the main term in this result matches Jarnik's mean square estimate (1.3) when $k = 3$, but by smoothing we expose an additional main term and a significant separation between the main terms and error term. An expression for the constant C_3 involves coefficients from the Laurent expansion of an L -function, and is harder to state exactly. Numerical approximation suggests that $C_3 \approx 10.6$.

For $k > 3$, it is possible to reduce the error term to $O_{\epsilon}(X^{k-2+\frac{3-k}{2}+\epsilon})$, although this introduces additional main terms with coefficients that are explicit but hard to compute. Due to a line of spectral poles in the Dirichlet

series $D(s, P_k \times P_k)$, which we will define below, we believe this result is the best smooth result possible.

The smoothed second moment in Theorem 1.1 can be thought of as a discrete Laplace transform. In [Ivi01], Ivić proved that

$$\int_0^\infty P_2(t)^2 e^{-t/X} dt = cX^{\frac{3}{2}} - X + O(X^{\frac{2}{3}+\epsilon})$$

for a known constant c , which can be thought of as a normal continuous Laplace transform of the lattice point discrepancy in dimension two. As an application of Theorem 1.1, we are able to prove a very strong result concerning the Laplace transform for dimensions $k \geq 3$.

Theorem 1.3. *For any $\epsilon > 0$, the smoothed second moment of the lattice point discrepancy for dimension $k \geq 3$ is given by*

$$\begin{aligned} \int_0^\infty P_k(t)^2 e^{-t/X} dt &= \delta_{[k=3]} C'_3 X^{k-1} (\log X + 1 - \gamma) + \delta_{[k=4]} C'_4 \Gamma(k - \tfrac{3}{2}) X^{k-\frac{3}{2}} \\ &\quad + C_k \Gamma(k-1) X^{k-1} - \frac{\Gamma(k-1) \pi^k}{6\Gamma(\frac{k}{2})^2} X^{k-1} + O(X^{k-2+\epsilon}), \end{aligned}$$

where the constants are the same as in Theorem 1.1.

Remark 1.4. As in Theorem 1.1, the techniques of this paper can be used to give further secondary terms and reduced error terms in dimensions $k > 3$.

By combining information from two more smoothed sums we are able to prove our main result, an analogue of Theorem 1.1 with a sharp cutoff.

Theorem 1.5. *For each $k \geq 3$ there exists a $\lambda > 0$ such that*

$$\sum_{n \leq X} P_k(n)^2 = \delta_{[k=3]} X^{k-1} \left(\frac{C'_3}{2} \log X - \frac{C'_3}{4} \right) + \frac{C_k}{k-1} X^{k-1} + O_\lambda(X^{k-1-\lambda}),$$

where C'_3 and C_k ($k \geq 3$) are the same constants as in Theorem 1.1.

Theorem 1.5 resembles the smoothed result (Theorem 1.1) up to constants, although the error bound is worse. Notice that in dimension three, Theorem 1.5 exhibits a second main term and additional power-savings in the error term.

The sum in Theorem 1.5 is closely related to Jarnik's mean square results (1.3) and (1.4). However, the two results differ in that Jarnik considers an integral over $[0, X]$, while we consider a sum of $P_k(n)$ over integral values up to X . For arithmetic applications, we believe that the sum is a more natural object of study than the integral. But as a corollary to Theorem 1.5, we are able to strengthen Jarnik's mean square estimate given in (1.3).

Theorem 1.6. *There exists $\lambda > 0$ such that*

$$\int_0^X (P_3(x))^2 dx = \frac{C'_3}{2} X^2 \log X + \left(\frac{C_3}{2} - \frac{C'_3}{4} - \frac{\pi^2}{3} \right) X^2 + O_\lambda(X^{2-\lambda}),$$

where C'_3 and C_3 are the same constants as in Theorem 1.1.

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DESCRIPTION OF METHODOLOGY AND OUTLINE OF PAPER

We approach this problem by understanding the analytic properties of the Dirichlet series associated to $S_k(n)^2$ and $P_k(n)^2$, defined by

$$D(s, S_k \times S_k) = \sum_{n=1}^{\infty} \frac{S_k(n)^2}{n^{s+k}}, \quad D(s, P_k \times P_k) = \sum_{n=1}^{\infty} \frac{P_k(n)^2}{n^{s+k-2}}.$$

Note that the k and $k-2$ in the exponents serve to normalize the Dirichlet series to converge absolutely for $\operatorname{Re} s > 1$, based on known mean square results. These two Dirichlet series are closely related to the series studied by the authors in [HKLDW17a, HKLDW17b], in which meromorphic continuations were given and studied for the Dirichlet series

$$\sum_{n \geq 1} \frac{S_f(n)^2}{n^s},$$

where $S_f(n) = \sum_{m \leq n} a(m)$ are partial sums of the coefficients of a $\mathrm{GL}(2)$ cusp form $f(z) = \sum a(n)e(nz)$. Indeed, the techniques and analysis in this paper build on the methodology introduced to study the cusp form case.

In §2, we show that the meromorphic properties of $D(s, P_k \times P_k)$ can be understood from the properties of $D(s, S_k \times S_k)$, and vice versa. We then decompose $D(s, S_k \times S_k)$ into *diagonal* and *off-diagonal* pieces. In §3.3 and §4 we prove that the pieces of $D(s, S_k \times S_k)$ have meromorphic continuations to the complex plane. This analysis culminates in Theorem 5.1, which gives that $D(s, S_k \times S_k)$ and $D(s, P_k \times P_k)$ have meromorphic continuation to the plane.

As in [HKLDW17a], the central challenge is determining the analytic behavior of the *off-diagonal*, which involves the shifted convolution sum

$$Z_k(s, w) = \sum_{h \geq 1} \sum_{n \geq 0} \frac{r_k(n+h)r_k(n)}{(n+h)^{s+\frac{k}{2}-1}h^w}.$$

Heuristically, this multiple Dirichlet series can be obtained from a Petersson inner product,

$$\langle |\theta^k|^2 \operatorname{Im}(\cdot)^{\frac{k}{2}}, P_h(\cdot, \bar{s}) \rangle,$$

where $P_h(z, s)$ is a Poincaré series and $\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$ is the standard theta function. In contrast to the cusp form case, however, $\theta(z)$ has moderate growth, complicating the spectral analysis of the inner product. Thus it

is necessary to modify $|\theta^k|^2$ to remove this growth. In §3 we subtract appropriate linear combinations of Eisenstein series evaluated at specific values such that the resulting function is square-integrable.

With this modification, in §6 we are able to use an inverse Mellin transform to extract information out of the meromorphic properties of $D(s, S_k \times S_k)$ and to prove the asymptotic behavior for the smoothed sum in Theorem 1.1.

Similar techniques are used to produce a sharp second moment in §7. This is achieved by using two different smooth cutoff transforms: a bump function of compact support and the same concentrating integral used by the authors in [HKLDW17b] to produce short interval estimates. In combination, these transforms prove Theorem 1.5.

In §8, we apply Theorem 1.1 to prove Theorem 1.3, our estimate for the Laplace transform of $P_k(t)^2$. The sum in Theorem 1.1 can be considered as an integral of a step function, and we study the difference between this integral and the continuous Laplace transform.

We apply similar techniques in §9 to prove our final result, a refinement of Jarnik's dimension three mean square result (1.3). Known bounds for $P_3(n)$ quickly reduce our study to bounds for the cross term

$$\sum_{n \leq X} P_3(n) n^{\frac{1}{2}}.$$

We extract a main term and power-savings error for this sum using the meromorphic properties of the Dirichlet series with coefficients $P_3(n)$ and an integral transform.

DIRECTIONS FOR FURTHER RESEARCH

As presented here, Theorems 1.5 and 1.6 show that there are two main terms and a power-savings error term in dimension three mean square estimates, but do not state the size of the power-savings in the error. In forthcoming work, the authors will analyze the growth properties of the Dirichlet series $D(s, S_k \times S_k)$ and $D(s, P_k \times P_k)$ and identify the size of the power-savings. In close analogy to [HKLDW17b], the analysis is delicate and the largest obstacle is obtaining a nuanced understanding of the growth properties of the Petersson inner product $\langle |\theta|^{2k} y^{\frac{k}{2}}, \mu_j \rangle$ for Maass forms μ_j . The authors conjecture that $\lambda = \frac{1}{5} - \epsilon$ is admissible in Theorem 1.5 (in dimension $k = 3$) and Theorem 1.6, for any $\epsilon > 0$. It is not clear what the optimal error bound should be.

The methodology used to prove Theorem 1.5 focused on the dimension three case, as this is the least understood. It may be possible to use the meromorphic properties of $D(s, S_k \times S_k)$ for $k \geq 4$ to prove improved estimates for higher dimensions as well. This is especially interesting in dimension four, as the smooth second moment in Theorem 1.1 suggests the existence

of a second main term in the sharp second moment of $P_4(n)$ which we have not been able to verify.

It may be possible to modify the techniques of this paper to approach the classical Gauss circle problem in two dimensions, or to understand the lattice point discrepancy problem for general ellipsoids. It would be interesting to understand the meromorphic properties of $D(s, P_2 \times P_2)$ and how it differs from the Dirichlet series associated to the Gauss circle problem in higher dimensions.

2. DECOMPOSITION OF $D(s, S_k \times S_k)$

Let V_k denote the volume of the k -sphere of radius one. Then $P_k(n)^2$ and $S_k(n)^2$ are related by the formula

$$P_k(n)^2 = S_k(n)^2 - 2V_k n^{\frac{k}{2}} S_k(n) + V_k^2 n^k. \quad (2.1)$$

This relationship induces a relationship between $D(s, P_k \times P_k)$ and $D(s, S_k \times S_k)$, described explicitly in the following proposition.

Proposition 2.1. *The Dirichlet series $D(s, P_k \times P_k)$ is related to $D(s, S_k \times S_k)$ through the equality*

$$\begin{aligned} D(s, P_k \times P_k) &= D(s - 2, S_k \times S_k) + V_k^2 \zeta(s - 2) \\ &\quad - 2V_k \zeta(s + \frac{k}{2} - 2) - 2V_k L(s - 1, \theta^k) \\ &\quad - \frac{2V_k}{2\pi i} \int_{(\sigma)} L(s - 1 - z, \theta^k) \zeta(z) \frac{\Gamma(z) \Gamma(s + \frac{k}{2} - 2 - z)}{\Gamma(s + \frac{k}{2} - 2)} dz, \end{aligned} \quad (2.2)$$

when $\sigma > 1$ and $\operatorname{Re} s > \sigma$, where $L(s, \theta^k)$ is the normalized L -function

$$L(s, \theta^k) := \sum_{n \geq 1} \frac{r_k(n)}{n^{s + \frac{k}{2} - 1}}$$

associated to the k -th power of the theta function $\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$.

Here and throughout this paper, we use the common notation

$$\frac{1}{2\pi i} \int_{(\sigma)} f(z) dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} f(\sigma + it) dt.$$

Proof. We begin with (2.1), divide each term by n^{s+k-2} , and sum over $n \geq 1$. The left-hand side and first term on the right-hand side are immediate from the definitions of $D(s, P_k \times P_k)$ and $D(s - 2, S_k \times S_k)$, respectively. Similarly, the third term on the right-hand side is immediately recognizable as $V_k^2 \zeta(s - 2)$.

For the second term, note that

$$S_k(n) = \sum_{m=0}^n r_k(m) = 1 + r_k(n) + \sum_{m=1}^{n-1} r_k(m).$$

Multiplying by $n^{\frac{k}{2}}$, dividing by n^{s+k-2} , and summing over $n \geq 1$ yields

$$\zeta(s + \frac{k}{2} - 2) + \sum_{n \geq 1} \frac{r_k(n)}{n^{s+\frac{k}{2}-2}} + \sum_{\substack{n \geq 1 \\ 0 < m < n}} \frac{r_k(m)}{n^{s+\frac{k}{2}-2}}.$$

Swapping the order of summation in the final sum and writing $n = m + h$ shows that

$$\sum_{n=1}^{\infty} \frac{S_k(n)}{n^{s+\frac{k}{2}-2}} = \zeta(s + \frac{k}{2} - 2) + L(s - 1, \theta^k) + \sum_{m, h \geq 1} \frac{r_k(m)}{(h + m)^{s+\frac{k}{2}-2}}. \quad (2.3)$$

We decouple m and h in the last sum with the identity

$$\frac{1}{(m + h)^s} = \frac{1}{2\pi i} \int_{(\sigma)} \frac{1}{m^{s-z} h^z} \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)} dz, \quad (\sigma > 0, \operatorname{Re} s > \sigma) \quad (2.4)$$

which follows from the Barnes integral 6.422(3) of [GR15]. For $\sigma > 1$, the h sum now converges absolutely and can be collected into a single $\zeta(z)$, and for $\operatorname{Re} s$ sufficiently large the m sum can be collected into $L(s - 1 - z, \theta^k)$. Multiplication by $-2V_k$ identifies this with the second term in (2.1), and simplification completes the proof. \square

Through (2.2) it is possible to pass analytic information from $D(s, S_k \times S_k)$ to $D(s, P_k \times P_k)$, and vice versa. To understand the meromorphic continuation of $D(s, S_k \times S_k)$, we first decompose the Dirichlet series $D(s, S_k \times S_k)$ into a sum of simpler functions. Our methodology is a variant of the methodology used in the proof of Proposition 3.1 in [HKLDW17a] and builds on the proof of the previous proposition, albeit with the added wrinkle of including shifted sums.

Proposition 2.2. *The Dirichlet series associated to $S_k(n)^2$ decomposes into*

$$D(s, S_k \times S_k) = \zeta(s + k) + W_k(s) + \frac{1}{2\pi i} \int_{(\sigma)} W_k(s - z) \zeta(z) \frac{\Gamma(z)\Gamma(s + k - z)}{\Gamma(s + k)} dz \quad (2.5)$$

for $\operatorname{Re} s > 2$ and $1 < \sigma < \operatorname{Re}(s - 1)$, in which

$$W_k(s) = \sum_{n=1}^{\infty} \frac{r_k(n)^2}{n^{s+k}} + 2Z_k(s + \frac{k}{2} + 1, 0),$$

$$Z_k(s, w) = \sum_{h \geq 1} \sum_{n \geq 0} \frac{r_k(n + h)r_k(n)}{(n + h)^{s+\frac{k}{2}-1} h^w}.$$

Here $Z_k(s, w)$ converges locally normally for $\operatorname{Re} s > 1 + \frac{k}{2}$ and $\operatorname{Re} w \geq 0$.

Proof. We may write

$$\begin{aligned} S_k(n)^2 &= \sum_{m \leq n} \sum_{\ell \leq n} r_k(m) r_k(\ell) = \sum_{m \leq n} r_k(m)^2 + 2 \sum_{\ell < m \leq n} r_k(m) r_k(\ell) \\ &= 1 + r_k(n)^2 + \sum_{0 < m < n} r_k(m)^2 + 2 \sum_{m < n} r_k(m) r_k(n) + 2 \sum_{\ell < m < n} r_k(m) r_k(\ell). \end{aligned}$$

In the second line, we separated out the terms in which $m = n$.

Dividing by n^{s+k} and summing over $n \geq 1$ gives

$$\begin{aligned} D(s, S_k \times S_k) &= \sum_{n=1}^{\infty} \frac{1}{n^{s+k}} + \left(\sum_{n=1}^{\infty} \frac{r_k(n)^2}{n^{s+k}} + 2 \sum_{\substack{n \geq 1 \\ m < n}} \frac{r_k(m) r_k(n)}{n^{s+k}} \right) \\ &\quad + \left(\sum_{\substack{n \geq 1 \\ 0 < m < n}} \frac{r_k(m)^2}{n^{s+k}} + 2 \sum_{\substack{n \geq 1 \\ \ell < m < n}} \frac{r_k(m) r_k(\ell)}{n^{s+k}} \right). \end{aligned}$$

We recognize the first term as a zeta function. The second and third terms represent the diagonal and off-diagonal (resp.) parts of a double summation, and we analyze them together. Swapping the order of summation and writing $n = m + h$ allows us to write the third term as

$$2 \sum_{\substack{n \geq 1 \\ m < n}} \frac{r_k(m) r_k(n)}{n^{s+k}} = 2 \sum_{\substack{m \geq 0 \\ h \geq 1}} \frac{r_k(m+h) r_k(m)}{(m+h)^{s+k}}.$$

We now recognize the second and third terms as $W_k(s)$.

The fourth and fifth terms are also closely related. Writing $n = m + h$ and swapping the order of summation allows us to write

$$\sum_{\substack{n \geq 1 \\ 0 < m < n}} \frac{r_k(m)^2}{n^{s+k}} + 2 \sum_{\substack{n \geq 1 \\ \ell < m < n}} \frac{r_k(m) r_k(\ell)}{n^{s+k}} = \sum_{\substack{h \geq 1 \\ m \geq 1}} \frac{r_k(m)^2}{(m+h)^{s+k}} + \sum_{\substack{h \geq 1 \\ m \geq 1 \\ \ell < m}} \frac{r_k(m) r_k(\ell)}{(m+h)^{s+k}}.$$

Notice that this pair of sums is exactly the same as the pair of sums in $W_k(s)$, except that the denominators are shifted by h and there is an additional h sum. We decouple the h from m by using the Barnes integral identity (2.4) again. For $\sigma > 1$, the h sum converges absolutely and can be collected into a zeta function. Simplification completes the proof of (2.5).

To see that $Z_k(s, w)$ converges locally normally in the range specified, it suffices by positivity to show that

$$Z_k(s, 0) = \sum_{h \geq 1} \sum_{n \geq 0} \frac{r_k(n+h) r_k(n)}{(n+h)^{s+\frac{k}{2}-1}} = \sum_{m \geq 1} \frac{r_k(m)}{m^{s+\frac{k}{2}-1}} \sum_{\ell < m} r_k(\ell)$$

converges absolutely for $\operatorname{Re} s > 1 + \frac{k}{2}$. In turn, this follows from the estimate $S_k(m) = O(m^{\frac{k}{2}})$ and absolute convergence of $L(s, \theta^k)$ in $\operatorname{Re} s > 1$. \square

3. MEROMORPHIC CONTINUATION OF $Z_k(s, w)$

In this section we follow a construction method analogous to that in [HH16, HKLDW17a], and we adapt the notation there. We seek to understand

$$Z_k(s, w) = \sum_{h \geq 1} \sum_{m \geq 0} \frac{r_k(m+h)r_k(m)}{(m+h)^{s+\frac{k}{2}-1}h^w}$$

by first fixing a single h and recognizing the remaining sum over m as a Petersson inner product of Poincaré series with an appropriate modular form, namely

$$\langle |\theta^k(\cdot)|^2 \operatorname{Im}(\cdot)^{\frac{k}{2}}, P_h(\cdot, \bar{s}) \rangle = \int_{\Gamma_0(4) \backslash \mathcal{H}} |\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}} \overline{P_h(z, \bar{s})} d\mu(z), \quad (3.1)$$

in which $d\mu(z) = dx dy / y^2$ and $P_h(z, s)$ is the Poincaré series

$$P_h(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \operatorname{Im}(\gamma z)^s e^{2\pi i h \gamma z}.$$

By expanding the inner product (3.1), we get

$$\langle |\theta^k(\cdot)|^2 \operatorname{Im}(\cdot)^{\frac{k}{2}}, P_h(\cdot, \bar{s}) \rangle = \frac{\Gamma(s + \frac{k}{2} - 1)}{(4\pi)^{s+\frac{k}{2}-1}} D_k(s; h),$$

where we define

$$D_k(s; h) = \sum_{m=0}^{\infty} \frac{r_k(m+h)r_k(m)}{(m+h)^{s+\frac{k}{2}-1}} \quad (3.2)$$

for $\operatorname{Re} s$ sufficiently large. Dividing by h^w and summing over $h \geq 1$ recovers $Z_k(s, w)$,

$$Z_k(s, w) = \sum_{h \geq 1} \frac{D_k(s; h)}{h^w} = \frac{(4\pi)^{s+\frac{k}{2}-1}}{\Gamma(s + \frac{k}{2} - 1)} \sum_{h \geq 1} \frac{\langle |\theta^k(\cdot)|^2 \operatorname{Im}(\cdot)^{\frac{k}{2}}, P_h(\cdot, \bar{s}) \rangle}{h^w}.$$

We would like to understand $Z_k(s, w)$ by expressing $\langle |\theta^k|^2 \operatorname{Im}^{\frac{k}{2}}, P_h \rangle$ in a different way, by replacing P_h with its spectral expansion. However, this is complicated by the fact that $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}}$ is not in $L^2(\Gamma_0(4) \backslash \mathcal{H})$, so it is necessary to modify $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}}$ to be square integrable. We accomplish this by subtracting Eisenstein series associated to the cusps of $\Gamma_0(4)$, chosen to cancel the polynomial growth of $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}}$.

3.1. Modifying $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}}$ to be square integrable. Let $E_{\mathfrak{a}}(z, s)$ denote the Eisenstein series attached to the cusp \mathfrak{a} for the group $\Gamma_0(4)$, given by

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(4)} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s,$$

where $\Gamma_{\mathfrak{a}} \subset \Gamma_0(4)$ is the stabilizer of the cusp \mathfrak{a} , and $\sigma_{\mathfrak{a}} \in \operatorname{PSL}_2(\mathbb{R})$ satisfies $\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$ and induces an isomorphism $\Gamma_{\mathfrak{a}} \cong \Gamma_\infty$ via conjugation. These

Eisenstein series have Fourier expansions, which (following [DI83]) can be written in the form

$$\begin{aligned} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, s) &= \delta_{[\mathfrak{a}=\mathfrak{b}]}y^s + \pi^{\frac{1}{2}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \varphi_{\mathfrak{a}\mathfrak{b}0}(s) y^{1-s} \\ &\quad + \frac{2\pi^s y^{\frac{1}{2}}}{\Gamma(s)} \sum_{n \neq 0} |n|^{s-\frac{1}{2}} \varphi_{\mathfrak{a}\mathfrak{b}n}(s) K_{s-\frac{1}{2}}(2\pi|n|y) e(nx) \end{aligned} \quad (3.3)$$

with known coefficients $\varphi_{\mathfrak{a}\mathfrak{b}n}(s)$. When $\mathfrak{b} = \infty$ we will often write these coefficients as $\varphi_{\mathfrak{a}n}(s)$. From (3.3) and asymptotics of the K -Bessel function it is clear that

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, \frac{k}{2}) = \delta_{[\mathfrak{a}=\mathfrak{b}]}y^{\frac{k}{2}} + \pi^{\frac{1}{2}} \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2})} \varphi_{\mathfrak{a}\mathfrak{b}0}(\frac{k}{2}) y^{1-\frac{k}{2}} + O_k(e^{-2\pi y}) \quad (3.4)$$

as $\text{Im } z \rightarrow \infty$. For $k \geq 3$, we conclude that $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z, \frac{k}{2})$ vanishes as $\text{Im } z \rightarrow \infty$ except in the case $\mathfrak{a} = \mathfrak{b}$, where it converges polynomially fast to $y^{\frac{k}{2}}$.

Lemma 3.1. *For $k \geq 3$, the function $\mathcal{V}(z)$ given by*

$$\mathcal{V}(z) := |\theta^k(z)|^2 \text{Im}(z)^{\frac{k}{2}} - E_{\infty}(z, \frac{k}{2}) - E_0(z, \frac{k}{2}),$$

vanishes at each of the cusps of $\Gamma_0(4)$. Therefore $\mathcal{V}(z) \in L^2(\Gamma_0(4) \backslash \mathcal{H})$.

Proof. We compute the growth of $|\theta^k(z)|^2 \text{Im}(z)^{\frac{k}{2}}$ at the three cusps $0, \frac{1}{2}$, and ∞ of $\Gamma_0(4)$ and compare to that of the Eisenstein series.

At the cusp ∞ , we observe directly from the Fourier expansion that

$$|\theta^k(z)|^2 \text{Im}(z)^{\frac{k}{2}} = y^{\frac{k}{2}} (1 + O(e^{-2\pi y}))$$

as $\text{Im } z \rightarrow \infty$. Thus growth at the ∞ cusp is exactly cancelled by subtracting the Eisenstein series $E_{\infty}(z, \frac{k}{2})$.

At the cusp 0 , we use $\sigma_0 = \begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$ to compute

$$\begin{aligned} \theta|_{\sigma_0}(z) &= (-2iz)^{-\frac{1}{2}} \theta \left(\begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix} z \right) = (-2iz)^{-\frac{1}{2}} \theta \left(-\frac{1}{4z} \right) \\ &= (-2iz)^{-\frac{1}{2}} (-2iz)^{\frac{1}{2}} \theta(z) = \theta(z), \end{aligned}$$

in which we've used the involution equation $\theta(-1/4z) = (-2iz)^{1/2} \theta(z)$ for the theta function. Therefore $|\theta^k(\sigma_0(z))|^2 \text{Im}(\sigma_0 z)^{\frac{k}{2}} = y^{\frac{k}{2}} (1 + O(e^{-2\pi y}))$ as $z \rightarrow \infty$, hence subtracting $E_0(z, \frac{k}{2})$ cancels the growth at the 0 cusp.

At the cusp $\frac{1}{2}$, we use $\sigma_{1/2} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ to compute

$$\begin{aligned} \theta|_{\sigma_{1/2}}(z) &= (2z+1)^{-\frac{1}{2}} \theta\left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} z\right) = (2z+1)^{-\frac{1}{2}} \theta\left(\frac{z}{2z+1}\right) \\ &= (2z+1)^{-\frac{1}{2}} \theta\left(\frac{-1}{-2-1/z}\right) = \left(\frac{i}{2z}\right)^{\frac{1}{2}} \theta\left(-\frac{1}{2} - \frac{1}{4z}\right) \\ &= \left(\frac{i}{2z}\right)^{\frac{1}{2}} \left(2\theta\left(-\frac{1}{z}\right) - \theta\left(-\frac{1}{4z}\right)\right), \end{aligned} \quad (3.5)$$

in which we've used that $\theta(z - \frac{1}{2}) = 2\theta(4z) - \theta(z)$, as can be seen by comparing the Fourier series of each term. Applying the involution equation to each theta function in (3.5), we see that $\theta|_{\sigma_{1/2}}(z) = \theta(\frac{z}{4}) - \theta(z)$, hence

$$\theta(\sigma_{\frac{1}{2}} z) \operatorname{Im}(\sigma_{\frac{1}{2}} z)^{\frac{1}{4}} = O\left(e^{-\pi y/2}\right)$$

as $z \rightarrow \infty$. Thus $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}} \rightarrow 0$ as $z \rightarrow \frac{1}{2}$ and it is not necessary to mitigate any growth at the cusp $\frac{1}{2}$. \square

We will use $\mathcal{V}(z)$ in place of $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}}$ to derive the analytic properties of $Z_k(s, w)$. Replacing (3.1) with the inner product $\langle \mathcal{V}(\cdot), P_h(\cdot, \bar{s}) \rangle$ and performing the calculations from the start of this section yields

$$\begin{aligned} &\frac{(4\pi)^{s+\frac{k}{2}-1}}{\Gamma(s+\frac{k}{2}-1)} \langle \mathcal{V}, P_h(\cdot, \bar{s}) \rangle \\ &= D_k(s; h) - \frac{(2\pi)^k \Gamma(s - \frac{k}{2})}{\Gamma(\frac{k}{2}) \Gamma(s)} \frac{(\varphi_{\infty h}(\frac{k}{2}) + \varphi_{0h}(\frac{k}{2}))}{h^{s-\frac{k}{2}}}, \end{aligned} \quad (3.6)$$

where $D_k(s; h)$ is as in (3.2). We note that we use [GR15, 6.621(3)] to evaluate the y -integral involved in evaluating this inner product in the Eisenstein series case. Dividing by h^w , summing over $h \geq 1$, and rearranging yields

$$\begin{aligned} Z_k(s, w) &= \frac{(4\pi)^{s+\frac{k}{2}-1}}{\Gamma(s+\frac{k}{2}-1)} \sum_{h \geq 1} \frac{\langle \mathcal{V}, P_h(\cdot, \bar{s}) \rangle}{h^w} \\ &\quad + \frac{(2\pi)^k \Gamma(s - \frac{k}{2})}{\Gamma(\frac{k}{2}) \Gamma(s)} \sum_{h \geq 1} \frac{(\varphi_{\infty h}(\frac{k}{2}) + \varphi_{0h}(\frac{k}{2}))}{h^{s+w-\frac{k}{2}}}. \end{aligned} \quad (3.7)$$

3.2. Spectral Expansion. By Selberg's Spectral Theorem (as in [IK04, Theorem 15.5]), the Poincaré series $P_h(z, s)$ has a spectral expansion of the form

$$\begin{aligned} P_h(z, s) &= \sum_j \langle P_h(\cdot, s), \mu_j \rangle \mu_j(z) \\ &\quad + \sum_{\mathfrak{a}} \frac{V}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot, s), E_{\mathfrak{a}}(\cdot, \frac{1}{2} + it) \rangle E_{\mathfrak{a}}(z, \frac{1}{2} + it) dt, \end{aligned} \quad (3.8)$$

where V is the volume of the fundamental domain for $\Gamma_0(4)\backslash\mathcal{H}$, \mathbf{a} ranges over the cusps of $\Gamma_0(4)\backslash\mathcal{H}$, and $\{\mu_j\}$ denotes an orthonormal basis of Maass forms for $L^2(\Gamma_0(4)\backslash\mathcal{H})$ with associated types $\frac{1}{2} + it_j$ which are also taken to be simultaneous Hecke eigenforms. We will refer to the sum over j as the “discrete part of the spectrum” and the sum of integrals of Eisenstein series as the “continuous part of the spectrum.” These Maass forms admit Fourier expansions,

$$\mu_j(z) = \sum_{n \neq 0} \rho_j(n) y^{\frac{1}{2}} K_{it_j}(2\pi|n|y) e(nx),$$

where $e(x) = e^{2\pi i x}$, and have associated L -functions of the form

$$L(s, \mu_j) = \sum_{n \geq 1} \frac{\rho_j(n)}{n^s}.$$

In this section, we use the spectral expansion (3.8) in the inner product in (3.7) to prove the following proposition.

Proposition 3.2. *For $\operatorname{Re} s$ sufficiently large, the shifted convolution sum $Z_k(s, w)$ can be expressed as*

$$\begin{aligned} Z_k(s, w) &= \frac{(2\pi)^k \Gamma(s - \frac{k}{2})}{\Gamma(\frac{k}{2}) \Gamma(s)} \sum_{h=1}^{\infty} \frac{(\varphi_{0h}(\frac{k}{2}) + \varphi_{\infty h}(\frac{k}{2}))}{h^{w+s-\frac{k}{2}}} \\ &\quad + \frac{(4\pi)^{\frac{k}{2}}}{2} \sum_j G(s, it_j) L(s + w - \frac{1}{2}, \mu_j) \langle \mathcal{V}, \mu_j \rangle \\ &\quad + \frac{(4\pi)^{\frac{k}{2}} V}{4\pi i} \sum_{\mathbf{a}} \int_{(0)} \frac{G(s, z) \pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2} + z)} \sum_{h \geq 1} \frac{\overline{\varphi_{\mathbf{a}h}(\frac{1}{2} - z)}}{h^{s+w-\frac{1}{2}-z}} \langle \mathcal{V}, E_{\mathbf{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle dz, \end{aligned} \quad (3.9)$$

in which $G(s, z)$ denotes the collected gamma factors,

$$G(s, z) := \frac{\Gamma(s - \frac{1}{2} + z) \Gamma(s - \frac{1}{2} - z)}{\Gamma(s + \frac{k}{2} - 1) \Gamma(s)}.$$

We refer to the first line of (3.9) as the “non-spectral part,” to the second line as the “discrete part of the spectrum,” and to the third line as the “continuous part of the spectrum.”

Proof. The inner product of μ_j against the Poincaré series gives

$$\langle P_h(\cdot, s), \mu_j \rangle = \frac{\overline{\rho_j(h)} \sqrt{\pi}}{(4\pi h)^{s-\frac{1}{2}}} \frac{\Gamma(s - \frac{1}{2} - it_j) \Gamma(s - \frac{1}{2} + it_j)}{\Gamma(s)}.$$

It follows that the discrete part of the spectrum of $P_h(z, s)$ can be written

$$\frac{\sqrt{\pi}}{(4\pi h)^{s-\frac{1}{2}} \Gamma(s)} \sum_j \overline{\rho_j(h)} \Gamma(s - \frac{1}{2} - it_j) \Gamma(s - \frac{1}{2} + it_j). \quad (3.10)$$

We have $\sup_j \{\operatorname{Im} t_j\} = 0$ as a consequence of Huxley's proof of the Selberg Eigenvalue Conjecture for Maass forms of small level [Hux85], which we note implies that (3.10) is analytic in the right half-plane $\operatorname{Re} s > \frac{1}{2}$.

The inner product of the Poincaré series against the Eisenstein series $E_a(z, w)$ can be computed to be

$$\langle P_h(\cdot, s), E_a(\cdot, w) \rangle = \frac{2\pi^{\bar{w}+\frac{1}{2}}}{(4\pi h)^{s-\frac{1}{2}}} h^{\bar{w}-\frac{1}{2}} \varphi_{ah}(\bar{w}) \frac{\Gamma(s+\bar{w}-1)\Gamma(s-\bar{w})}{\Gamma(s)\Gamma(\bar{w})},$$

provided that $\operatorname{Re} s > |\operatorname{Re} w - \frac{1}{2}| + \frac{1}{2}$. With $t \in \mathbb{R}$ and $w = \frac{1}{2} + it$, this specializes to

$$\langle P_h(\cdot, s), E_a(\cdot, \frac{1}{2} + it) \rangle = \frac{2\pi^{1-it} \varphi_{ah}(\frac{1}{2} - it)}{(4\pi h)^{s-\frac{1}{2}}} \frac{\Gamma(s - \frac{1}{2} - it)\Gamma(s - \frac{1}{2} + it)}{h^{it}\Gamma(s)\Gamma(\frac{1}{2} - it)},$$

which is valid provided that $\operatorname{Re} s > \frac{1}{2}$. Thus the continuous part of the spectrum of $P_h(z, s)$ takes the form

$$\frac{V}{2} \sum_a \int_{-\infty}^{\infty} \frac{\varphi_{ah}(\frac{1}{2} - it)\Gamma(s - \frac{1}{2} - it)\Gamma(s - \frac{1}{2} + it)}{(4\pi h)^{s-\frac{1}{2}}(\pi h)^{it}\Gamma(s)\Gamma(\frac{1}{2} - it)} E_a(z, \frac{1}{2} + it) dt. \quad (3.11)$$

Substituting the discrete part of the spectrum (3.10) and continuous part of the spectrum (3.11) into the expansion of the Poincaré series (3.8) gives

$$\begin{aligned} \langle \mathcal{V}, P_h(\cdot, \bar{s}) \rangle &= \frac{\sqrt{\pi}}{(4\pi h)^{s-\frac{1}{2}}\Gamma(s)} \sum_j \rho_j(h) \Gamma(s - \frac{1}{2} + it_j) \Gamma(s - \frac{1}{2} - it_j) \langle \mathcal{V}, \mu_j \rangle \\ &+ \frac{V}{2} \sum_a \int_{-\infty}^{\infty} \frac{\varphi_{ah}(\frac{1}{2} - it)\Gamma(s - \frac{1}{2} + it)\Gamma(s - \frac{1}{2} - it)}{(4\pi h)^{s-\frac{1}{2}}(\pi h)^{-it}\Gamma(s)\Gamma(\frac{1}{2} + it)} \langle \mathcal{V}, E_a(\cdot, \frac{1}{2} + it) \rangle dt. \end{aligned}$$

Finally, substituting into (3.7) and simplifying completes the proof. \square

3.3. Meromorphic Continuation. In order to provide meromorphic continuation of $Z_k(s, w)$, we give the meromorphic continuation of each part of (3.9). We will prove the following lemma as a step towards understanding the analytic behavior of $W_k(s)$, which we study in §4.

Lemma 3.3. *The shifted convolution $Z_k(s, w)$ has meromorphic continuation to \mathbb{C}^2 . In particular, the specialized convolution sum $Z_k(s, 0)$ has meromorphic continuation to the plane. For $\operatorname{Re} s > -\frac{1}{2}$, all poles of $Z_k(s, 0)$ come from the non-spectral part (which has poles at $s = 1 + \frac{k}{2} - j$ for $j \in \mathbb{Z}_{\geq 0}$) and the continuous part of the spectrum (whose poles appear within the residual terms \mathcal{R}_j^\pm , as defined in §3.3.3).*

3.3.1. Non-Spectral Part. When $\mathfrak{b} = \infty$ and the cusp \mathfrak{a} is represented in the form $\mathfrak{a} = u/v$ with $(u, v) = 1$, the exact definition of the coefficients $\varphi_{\mathfrak{a}\mathfrak{b}h}(t)$

in (3.3) is given by [DI83] as

$$\varphi_{ah}(t) = \left(\frac{(v, 4/v)}{4v} \right)^t \sum_{(\gamma, 4/v)=1}^{\infty} \gamma^{-2t} \sum_{\substack{\delta(\gamma v)^* \\ \gamma \delta \equiv u \pmod{(v, 4/v)}}} e\left(\frac{h\delta}{\gamma v}\right).$$

We represent the three inequivalent cusps $0, \frac{1}{2},$ and ∞ of $\Gamma_0(4)$ as $1, \frac{1}{2},$ and $\frac{1}{4},$ respectively. It is a standard exercise to compute these coefficients (see [Gol15, §3.1] for a similar calculation), and we find that

$$\begin{aligned} \varphi_{0h}(t) &= \frac{\sigma_{1-2t}^{(2)}(h)}{4^t \zeta^{(2)}(2t)}, & \varphi_{\frac{1}{2}h}(t) &= \frac{(-1)^h \sigma_{1-2t}^{(2)}(h)}{4^t \zeta^{(2)}(2t)}, \\ \varphi_{\infty h}(t) &= \frac{2^{2-4t} \sigma_{1-2t}(\frac{h}{4}) - 2^{1-4t} \sigma_{1-2t}(\frac{h}{2})}{\zeta^{(2)}(2t)}. \end{aligned}$$

in which $\zeta^{(2)}(t)$ is the Riemann zeta function with its 2-factor removed, $\sigma_{\nu}(h)$ is the sum of divisors function, and $\sigma_{\nu}^{(2)}(h)$ is the sum of odd-divisors function. Dividing by h^w and summing over h , we compute

$$\begin{aligned} \sum_{h \geq 1} \frac{\varphi_{0h}(t)}{h^w} &= \frac{\zeta(w) \zeta^{(2)}(w-1+2t)}{4^t \zeta^{(2)}(2t)}, \\ \sum_{h \geq 1} \frac{\varphi_{\frac{1}{2}h}(t)}{h^w} &= \frac{(2^{1-w} - 1) \zeta(w) \zeta^{(2)}(w-1+2t)}{4^t \zeta^{(2)}(2t)}, \\ \sum_{h \geq 1} \frac{\varphi_{\infty h}(t)}{h^w} &= \frac{\zeta(w) \zeta^{(2)}(w-1+2t)}{2^{4t} \zeta^{(2)}(2t)} \left(\frac{1}{4^{w-1}} - \frac{1}{2^{w-1}} \right). \end{aligned} \tag{3.12}$$

Applying these expressions to the spectral decomposition from Proposition 3.2, we rewrite the non-spectral part as

$$\frac{\pi^k \Gamma(s - \frac{k}{2}) \zeta(s + w - \frac{k}{2}) \zeta(s + w + \frac{k}{2} - 1)}{\Gamma(\frac{k}{2}) \Gamma(s) \zeta^{(2)}(k)} \left(1 + \frac{4}{2^{2s+2w}} - \frac{4}{2^{\frac{k}{2}+s+w}} \right).$$

This expression is analytic in the region $\operatorname{Re} s > k/2$ and $\operatorname{Re}(s+w) > 1+k/2$, and extends meromorphically to all of \mathbb{C}^2 with polar lines at $s+w = 1+k/2$, $s+w = 2-k/2$, and poles in s at poles of $\Gamma(s - \frac{k}{2})/\Gamma(s)$. Specializing to the case $w = 0$, we note potential poles at $s = 1 + \frac{k}{2} - j$ for each integer $j \geq 0$.

3.3.2. Discrete Part of the Spectrum. The discrete part of the spectrum from (3.9) has clear meromorphic continuation induced by the meromorphic continuations of the individual $L(s, \mu_j)$. We note that for any fixed s , the gamma functions in $G(s, it_j)$ give exponential decay so that the sum converges absolutely.

Following observations analogous to those in [HKLDW17a, §4.2], careful inspection of the inner product $\langle \mathcal{V}, \mu_j \rangle$ shows that $\langle \mathcal{V}, \mu_j \rangle = 0$ if μ_j is odd. Indeed, $|\theta^k(z)|^2 \operatorname{Im}(z)^{\frac{k}{2}}$ is even and Eisenstein series are orthogonal to cusp forms. Otherwise, if μ_j is even, we note by the functional equation of even

Maass forms that $L(-2m \pm it_j, \mu_j) = 0$ for any $m \in \mathbb{Z}_{\geq 0}$. Specializing now to $w = 0$, these two observations combine to indicate that the apparent poles at $s = \frac{1}{2} \pm it_j$ do not exist. Therefore the discrete part of the spectrum is analytic for $\operatorname{Re} s > -\frac{1}{2}$ and has poles at $s - \frac{1}{2} \pm it_j = -m$ for m odd, $m \in \mathbb{Z}_{>0}$.

3.3.3. Continuous Part of the Spectrum. The continuous part of the spectrum from (3.9) requires more nuanced analysis than the discrete part or non-spectral part, due to the interaction of independent complex variables.

For notational simplicity, we write the continuous part in the form

$$\frac{(4\pi)^{\frac{k}{2}} V}{4\pi i} \sum_{\mathfrak{a}} \int_{(0)} \frac{G(s, z) \pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2} + z)} \zeta_{\mathfrak{a}}(s + w, z) \langle \mathcal{V}, E_{\mathfrak{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle dz, \quad (3.13)$$

in which $\zeta_{\mathfrak{a}}(s, z)$ is defined by

$$\zeta_{\mathfrak{a}}(s + w, z) = \sum_{h \geq 1} \frac{\overline{\varphi_{ah}(\frac{1}{2} - z)}}{h^{s+w-\frac{1}{2}-z}}.$$

For better reference, we write these explicitly at each cusp via (3.12) as

$$\begin{aligned} \zeta_0(s + w, z) &= \frac{\zeta(s + w - \frac{1}{2} - z) \zeta^{(2)}(s + w - \frac{1}{2} + z)}{2^{1+2z} \zeta^{(2)}(1 + 2z)}, \\ \zeta_{\frac{1}{2}}(s + w, z) &= \frac{\zeta(s + w - \frac{1}{2} - z) \zeta^{(2)}(s + w - \frac{1}{2} + z)}{2^{1+2z} \zeta^{(2)}(1 + 2z)} \left(\frac{2^z}{2^{s+w-\frac{3}{2}}} - 1 \right), \\ \zeta_{\infty}(s + w, z) &= \frac{\zeta(s + w - \frac{1}{2} - z) \zeta(s + w - \frac{1}{2} + z)}{2^{2+4z} \zeta^{(2)}(1 + 2z)} \left(\frac{4^z}{4^{s+w-\frac{3}{2}}} - \frac{2^z}{2^{s+w-\frac{3}{2}}} \right). \end{aligned}$$

It is clear that the continuous part of the spectrum is analytic in the region $\operatorname{Re}(s + w) > \frac{3}{2}$ and $\operatorname{Re} s > \frac{1}{2}$, and that the integrand has apparent poles when $s + w - \frac{1}{2} \pm z = 1$ and $s = \frac{1}{2} \pm z - j$ for $j \in \mathbb{Z}_{\geq 0}$. It is now necessary to disentangle these poles from the integration variable.

Arguing as in [HKLDW17a, §4.4.2] and [HH16], we iteratively extend the meromorphic continuation of the continuous part of the spectrum by carefully shifting lines of integration and collecting residual terms.

For small $\epsilon > 0$, let $\operatorname{Re} s$ lie in the interval $(\frac{3}{2} - \operatorname{Re} w, \frac{3}{2} - \operatorname{Re} w + \epsilon)$ and furthermore suppose s is at least a distance of 2ϵ from the potential poles of $G(s, z)$. We shift the z -contour to the right, along a contour C which bends to remain in the zero-free region of $\zeta(1 - 2z)$ and thus avoids potential poles. In so doing, we pass a pole at $s + w - \frac{1}{2} - z = 1$ with residue

$$\mathcal{R}_1^- := \frac{(4\pi)^{\frac{k}{2}} V}{2} \operatorname{Res}_{z=s+w-\frac{3}{2}} \frac{G(s, z) \pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2} + z)} \sum_{\mathfrak{a}} \zeta_{\mathfrak{a}}(s + w, z) \langle \mathcal{V}, E_{\mathfrak{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle.$$

The 2-factors in $\zeta_{\infty}(s + w, z)$ and $\zeta_{\frac{1}{2}}(s + w, z)$ create zeros that cancel the pole, so the only cusp that gives a polar contribution at $z = s + w - \frac{3}{2}$ is the

0 cusp. Simplifying, we find that

$$\mathcal{R}_1^- = -\frac{(4\pi)^{\frac{k}{2}}V}{2\pi^{1-s-w}} \frac{\Gamma(1-w)\Gamma(2s+w-2) \langle \mathcal{V}, E_0(\cdot, 2-\bar{s}-\bar{w}) \rangle}{2^{2s+2w-2}\Gamma(s)\Gamma(s+\frac{k}{2}-1)\Gamma(s+w-1)}. \quad (3.14)$$

The residue $\mathcal{R}_1^- = \mathcal{R}_1^-(s, w)$ is analytic in the vertical strip $\operatorname{Re} s \in (1 - \operatorname{Re} w, \frac{3}{2} - \operatorname{Re} w + \epsilon)$ and has a straightforward meromorphic continuation to \mathbb{C}^2 . Our deformation of the contour integral (3.13) is analytic for s to the right of the contour $\frac{3}{2} - \operatorname{Re} w - C$ and to the left of the line $\frac{3}{2} - \operatorname{Re} w + \epsilon$. When s is moved just to the left of the $\frac{3}{2} - \operatorname{Re} w$ line in this region, we can shift the contour of z integration back to $\operatorname{Re} z = 0$. This passes over the *other* pole at $s + w - \frac{1}{2} + z = 1$ from the other zeta function and introduces a residue

$$\mathcal{R}_1^+ := \frac{(4\pi)^{\frac{k}{2}}V}{2} \operatorname{Res}_{z=\frac{3}{2}-s-w} \frac{G(s, z)\pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2}+z)} \sum_{\mathfrak{a}} \zeta_{\mathfrak{a}}(s+w, z) \langle \mathcal{V}, E_{\mathfrak{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle. \quad (3.15)$$

The residue \mathcal{R}_1^+ is also analytic for $\operatorname{Re} s \in (1 - \operatorname{Re} w, \frac{3}{2} - \operatorname{Re} w)$ and has a straightforward meromorphic continuation. We note that the shifted contour integral has no further poles with $\operatorname{Re}(s+w) > \frac{1}{2}$ or with $\operatorname{Re} s > \frac{1}{2}$. Therefore the continuous part of the spectrum, originally defined for $\operatorname{Re}(s+w) > \frac{3}{2}$ and $\operatorname{Re} s > \frac{1}{2}$, has meromorphic extension to $\operatorname{Re}(s+w) > \frac{1}{2}$ and $\operatorname{Re} s > \frac{1}{2}$, given by

$$\frac{(4\pi)^{\frac{k}{2}}V}{4\pi i} \sum_{\mathfrak{a}} \int_{(0)} \frac{G(s, z)\pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2}+z)} \zeta_{\mathfrak{a}}(s+w, z) \langle \mathcal{V}, E_{\mathfrak{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle dz + \mathcal{R}_1^+ - \mathcal{R}_1^-,$$

where by a slight abuse of notation we claim that the two residual terms $\mathcal{R}_1^{\pm}(s, w)$ appear in the continuation only when $\operatorname{Re}(s+w) < \frac{3}{2}$. Since the two residual terms are analytic for $\operatorname{Re}(s+w) > 1$, we have moreover an *analytic* continuation of the continuous part of the spectrum into $\operatorname{Re}(s+w) > 1$ and $\operatorname{Re} s > \frac{1}{2}$, past the apparent poles at $\operatorname{Re}(s+w) = \frac{3}{2}$.

We now iterate this argument to push the meromorphic continuation of the continuous part past additional polar lines, as in [HH16, §4, p. 481-483] or [HKLDW17a, §4]. That is, for $\operatorname{Re} s$ near $\frac{1}{2} - j$ with $j \in \mathbb{Z}_{\geq 0}$, we shift the line of integration in z past a pole due to a gamma factor in the numerator of $G(s, z)$, move s left past the polar line, and shift the line of integration back to the imaginary axis, passing a pole from the other gamma factor in the numerator of $G(s, z)$. Each iteration contributes two additional residual terms with opposite signs, denoted by $\mathcal{R}_{-j}^+ - \mathcal{R}_{-j}^-$, in which

$$\begin{aligned} \mathcal{R}_{-j}^+ &= \frac{(4\pi)^{\frac{k}{2}}V}{2} \sum_{\mathfrak{a}} \operatorname{Res}_{z=\frac{1}{2}-j-s} \frac{G(s, z)\pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2}+z)} \zeta_{\mathfrak{a}}(s+w, z) \langle \mathcal{V}, E_{\mathfrak{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle, \\ \mathcal{R}_{-j}^- &= \frac{(4\pi)^{\frac{k}{2}}V}{2} \sum_{\mathfrak{a}} \operatorname{Res}_{z=s+j-\frac{1}{2}} \frac{G(s, z)\pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2}+z)} \zeta_{\mathfrak{a}}(s+w, z) \langle \mathcal{V}, E_{\mathfrak{a}}(\cdot, \frac{1}{2} - \bar{z}) \rangle. \end{aligned}$$

Each of these residue terms has an easily understood meromorphic continuation. In this way, we obtain the meromorphic continuation of $Z_k(s, w)$ to the entire complex plane.

4. ANALYTIC BEHAVIOR OF $W_k(s)$

In this section, we outline some of the analytic properties of $W_k(s)$. These properties will be used in §5 to understand $D(s, P_k \times P_k)$.

Recall from Proposition 2.2 that

$$W_k(s) = \sum_{n \geq 1} \frac{r_k(n)^2}{n^{s+k}} + 2Z_k(s + \frac{k}{2} + 1, 0). \quad (4.1)$$

We refer to the sum in (4.1) as the *diagonal part*. The second term, $Z_k(s, w)$, is the *off-diagonal part*, which we recall decomposes into three terms we have called the *non-spectral*, *discrete*, and *continuous parts*. We address the meromorphic behavior of each part in turn, and produce the following theorem by assembling these parts together.

Theorem 4.1. *The function $W_k(s)$ has meromorphic continuation to all $s \in \mathbb{C}$. In the half-plane $\operatorname{Re} s > -\frac{k+3}{2}$, all but one of the poles of $W_k(s)$ occur at non-positive even integers, coming from the non-spectral part*

$$\mathfrak{E}_k(s) = \frac{2\pi^k \Gamma(s+1) \zeta(s+1) \zeta(s+k)}{\Gamma(\frac{k}{2}) \Gamma(s + \frac{k}{2} + 1) \zeta^{(2)}(k)} \left(1 + \frac{1}{2^{2s+k}} - \frac{1}{2^{s+k-1}} \right).$$

The function $W_k(s)$ has an additional pole at $s = -\frac{k+1}{2}$. When $k > 3$, this pole is simple and has residue

$$\operatorname{Res}_{s=-\frac{k+1}{2}} W_k(s) = (4\pi)^{\frac{k}{2}} V \frac{\langle \mathcal{V}, E_0(\cdot, \frac{3}{2}) \rangle}{\pi^{\frac{3}{2}} \Gamma(\frac{k-1}{2})}.$$

When $k = 3$, this pole is a double pole, and the Laurent series of $W_3(s)$ about $s = -2$ has principal part

$$-\frac{\pi^2}{3\zeta^{(2)}(3)(s+2)^2} + \frac{24a_0 V \zeta^{(2)}(3) - \pi^2 \gamma - \pi^2 \log(4\pi)}{3\zeta^{(2)}(3)(s+2)},$$

where a_0 is the constant term in the Laurent series for the meromorphic continuation of $\langle \mathcal{V}, E_0(\cdot, \frac{3}{2}) \rangle$ at $s = \frac{3}{2}$.

4.1. Diagonal Part. We recognize the diagonal part in terms of the Rankin–Selberg L -function associated to $\theta^k \times \theta^k$, written $L(s, \theta^k \times \theta^k)$ and defined by

$$L(s, \theta^k \times \theta^k) = \zeta(2s) \sum_{n \geq 1} \frac{r_k(n)^2}{n^{s+\frac{k}{2}-1}}.$$

As $y^{\frac{k}{2}}|\theta^k(z)|^2$ is not of rapid decay, we interpret this L -function through Gupta's generalization of Zagier's regularization method to congruence subgroups [Zag81, DG00]. This employs a regularized Rankin–Selberg transform of \mathcal{V} ,

$$\frac{1}{V} \int_0^\infty (c_0(y) - \psi_0(y)) y^{s-1} \frac{dy}{y} = \frac{\Gamma(s + \frac{k}{2} - 1)}{(4\pi)^{s + \frac{k}{2} - 1} V} \sum_{m=1}^\infty \frac{r_k(m)^2}{m^{s + \frac{k}{2} - 1}}, \quad (4.2)$$

where $c_0(y)$ is the constant Fourier coefficient of $\mathcal{V}(\sigma_0 z)$ and $\psi_0(y)$ is a linear combination of products of powers of $\log y$ and y for which $\mathcal{V}(\sigma_0 z) = \psi_0(y) + O(y^{-N})$ for all $N > 0$ as $\text{Im } z \rightarrow \infty$.

Here, $\psi_0(y)$ consists only of the constant multiple of $y^{1 - \frac{k}{2}}$ which appears in the constant coefficient of the $E_0(z, \frac{k}{2})$ Eisenstein series, as shown in (3.3). Thus by [DG00], the representation (4.2) can be identified as $\langle \mathcal{V}(\sigma_0 \cdot), E_0(\cdot, \bar{s}) \rangle$ for s where $1 - \frac{k}{2} < \text{Re } s < \frac{k}{2}$; has meromorphic continuation to the plane with potential poles at $s = \frac{k}{2}, 1, 0, 1 - \frac{k}{2}$, and at zeroes of $\zeta(2s)$; and satisfies a functional equation of shape $s \mapsto 1 - s$. Correspondingly, the diagonal part of $W_k(s)$ has potential poles at $s = -1, -\frac{k}{2}, -\frac{k}{2} - 1, -k$, and at zeroes of $\zeta(2s + k + 2)$.

For the leading pole at $s = -1$, we have

$$\text{Res}_{s=-1} \sum_{m=1}^\infty \frac{r_k(m)^2}{m^{s+k}} = \lim_{X \rightarrow \infty} \frac{k-1}{X^{k-1}} \sum_{m \leq X} r_k(m)^2 = \frac{\pi^k \zeta(k-1)}{\zeta^{(2)}(k) \Gamma(\frac{k}{2})^2}. \quad (4.3)$$

The second equality is the subject of [CKO05], who apply a general method for evaluating sums of positive definite quadratic forms due to Müller [Mül92]. The second pole occurs at $s = -\frac{k}{2}$ and can be understood through (4.2) to give the residue

$$\frac{(4\pi)^{\frac{k}{2}} V}{\Gamma(\frac{k}{2})} \text{Res}_{s=1} \langle \mathcal{V}, E_0(\cdot, \bar{s}) \rangle. \quad (4.4)$$

The poles from zeroes of the zeta function and the two remaining poles in the diagonal part can be analyzed using the functional equation for $L(s, \theta^k \times \theta^k)$, but these details will not be necessary as we will show that the diagonal part identically cancels with $\mathcal{R}_0^+ - \mathcal{R}_0^-$ in a region containing these poles.

Remark 4.2. It is possible to represent the diagonal part as a Rankin–Selberg transform of \mathcal{V} against either $E_0(z, s)$ or $E_\infty(z, s)$. By choosing the Eisenstein series $E_0(z, s)$ instead of $E_\infty(z, s)$ we are able to show cancellation between $L(s, \theta^k \times \theta^k)$ and $Z_k(s, 0)$ in a straightforward manner in §5.

4.2. Discrete Part. Examination of §3.3 reveals that the discrete part of $W_k(s)$ is analytic for $\text{Re } s > -\frac{k+3}{2}$, where we focus our analysis. The boundary of this region, the line $\text{Re } s = -\frac{k+3}{2}$, hosts a line of poles coming from the eigenvalues t_j of Maass forms.

4.3. Continuous Part. We now discuss the analytic properties of the continuous part of $W_k(s)$ in the right half-plane $\operatorname{Re} s > -\frac{k+3}{2}$. As shown in §3.3, many residual terms \mathcal{R}_{-j}^\pm appear in the meromorphic continuation of the continuous part as $\operatorname{Re} s$ decreases. However, the only residual terms present in $\operatorname{Re} s > -\frac{k+3}{2}$ are \mathcal{R}_1^\pm and \mathcal{R}_0^\pm .

In analogy with [HKLDW17a], we expect that $\mathcal{R}_1^+ = -\mathcal{R}_1^-$ when $w = 0$. This is correct, but is harder to prove in our current situation because \mathcal{R}_1^+ and \mathcal{R}_1^- occur as sums over the three cusps of $\Gamma_0(4)\backslash\mathcal{H}$.

Lemma 4.3. *With the notation of §3.3, we have*

$$\mathcal{R}_1^+(s, 0) = -\mathcal{R}_1^-(s, 0).$$

Proof. Beginning with the formula for \mathcal{R}_1^- given in (3.14), set $w = 0$ and apply the Gauss duplication formula to obtain

$$\mathcal{R}_1^-(s, 0) = -\frac{(4\pi)^{\frac{k}{2}}V}{2} \cdot \frac{\Gamma(s - \frac{1}{2})\pi^{s-\frac{3}{2}} \langle \mathcal{V}, E_0(\cdot, 2 - \bar{s}) \rangle}{2\Gamma(s)\Gamma(s + \frac{k}{2} - 1)}.$$

Let $\mathcal{E}(z, s) = \{E_a(z, s)\}_a$. Following Iwaniec [Iwa02], we have $\mathcal{E}(z, s) = \Phi(s)\mathcal{E}(z, 1-s)$, in which $\Phi(s)$ is the symmetric scattering matrix

$$\Phi(s) = \pi^{\frac{1}{2}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} (\varphi_{ab0}(s))_{a,b}$$

composed of the constant Fourier coefficients of the various Eisenstein series $E_a(\sigma_b z, s)$. In particular, we have that

$$\begin{aligned} E_0(z, s) = \frac{\sqrt{\pi}\Gamma(s - \frac{1}{2})}{\Gamma(s)\zeta^{(2)}(2s)} & \left(\frac{\zeta^{(2)}(2s-1)E_\infty(z, 1-s)}{4^s} \right. \\ & \left. + \frac{\zeta^{(2)}(2s-1)E_{\frac{1}{2}}(z, 1-s)}{4^s} + \frac{\zeta(2s-1)E_0(z, 1-s)}{2^{4s-1}} \right). \end{aligned}$$

We apply the Gauss duplication formula and the functional equations of $E_0(z, s)$ and the Riemann zeta function to transform \mathcal{R}_1^- into

$$\begin{aligned} & -\frac{(4\pi)^{\frac{k}{2}}\Gamma(2s-2)\pi^{2-s}\zeta(2s-2)}{2\Gamma(s)\Gamma(s + \frac{k}{2} - 1)\Gamma(2-s)\zeta^{(2)}(4-2s)} \times \left(\frac{\langle \mathcal{V}, E_0(\cdot, \bar{s}-1) \rangle}{2^{5-2s}} \right. \\ & \left. + \frac{(4^{3-2s} - 2^{3-2s})\langle \mathcal{V}, E_\infty(\cdot, \bar{s}-1) \rangle}{2^{8-4s}} + \frac{(2^{3-2s} - 1)\langle \mathcal{V}, E_{\frac{1}{2}}(\cdot, \bar{s}-1) \rangle}{2^{5-2s}} \right). \end{aligned}$$

Simplification shows this is equal to $-\mathcal{R}_1^+$. \square

The contribution from $\mathcal{R}_1^+(s, 0) - \mathcal{R}_1^-(s, 0)$, written with arguments as they appear within the term $2Z_k(s + \frac{k}{2} + 1, 0)$, thus takes the form

$$4\mathcal{R}_1^+(s + \frac{k}{2} + 1, 0) = (4\pi)^{\frac{k}{2}}V \frac{\Gamma(s + \frac{k}{2} + \frac{1}{2})\pi^{s+\frac{k-1}{2}} \langle \mathcal{V}, E_0(\cdot, 1 - \frac{k}{2} - \bar{s}) \rangle}{\Gamma(s + \frac{k}{2} + 1)\Gamma(s + k)}.$$

This term has infinitely many poles (at least, when k is even), of which at most two lie in the right half-plane $\operatorname{Re} s > -\frac{k+3}{2}$. There is a pole at $s = -\frac{k}{2}$ coming from the Eisenstein series, with residue

$$\operatorname{Res}_{s=-\frac{k}{2}} 4\mathcal{R}_1^+(s + \frac{k}{2} + 1, 0) = -\frac{(4\pi)^{\frac{k}{2}} V}{\Gamma(\frac{k}{2})} \operatorname{Res}_{s=1} \langle \mathcal{V}, E_0(\cdot, \bar{s}) \rangle.$$

A second pole appears at $s = 1 - k$ from the inner product (although not from the Eisenstein series), which is relevant to our study in the cases $k \leq 4$. In the case $k = 4$, the pole at $s = 1 - k$ in the inner product is cancelled by a zero in $\Gamma(s + \frac{k}{2} + 1)^{-1}$, and does not appear. In the remaining case $k = 3$ this pole collides with a pole at $s = -\frac{k+1}{2}$ coming from the gamma factor, creating a double pole with principal part

$$-\frac{\pi^2}{3\zeta^{(2)}(3)(s+2)^2} + \frac{24a_0 V \zeta^{(2)}(3) - \pi^2 \gamma - \pi^2 \log(4\pi)}{3\zeta^{(2)}(3)(s+2)},$$

in which γ is the Euler-Mascheroni constant and a_0 is the constant coefficient of the Laurent expansion of $\langle \mathcal{V}, E_0(\cdot, \bar{s}) \rangle$ about $s = \frac{3}{2}$.

For $k \geq 4$, the gamma factor pole at $s = -\frac{k+1}{2}$ is simple, with residue

$$\operatorname{Res}_{s=-\frac{k+1}{2}} 4\mathcal{R}_1^+(s + \frac{k}{2} + 1, 0) = (4\pi)^{\frac{k}{2}} V \frac{\langle \mathcal{V}, E_0(\cdot, \frac{3}{2}) \rangle}{\pi^{\frac{3}{2}} \Gamma(\frac{k-1}{2})}.$$

Further analogy with [HKLDW17a] leads us to expect that $\mathcal{R}_0^+(s, 0) = -\mathcal{R}_0^-(s, 0)$ and that $2\mathcal{R}_0^+(s, 0)$ shows significant cancellation with the diagonal term. A computation very similar to that performed in Lemma 4.3 shows that this is indeed the case.

Lemma 4.4. *With the notation of §3.3, we have*

$$\mathcal{R}_0^+(s, 0) = -\mathcal{R}_0^-(s, 0).$$

Simplifying $2\mathcal{R}_0^+(s, 0)$ gives

$$2\mathcal{R}_0^+(s, 0) = -\frac{V}{2} \cdot \frac{(4\pi)^{s+\frac{k}{2}-1}}{\Gamma(s+\frac{k}{2}-1)} \langle \mathcal{V}, E_\infty(\cdot, \bar{s}) \rangle.$$

As in §4.1, Zagier regularization identifies this expression with a Rankin–Selberg L -function,

$$\mathcal{R}_0^+ - \mathcal{R}_0^- = -\frac{1}{2} \sum_{m=1}^{\infty} \frac{r_k(m)^2}{m^{s+\frac{k}{2}-1}},$$

and we conclude that the second residual pair in the meromorphic continuation of $2Z_k(s + \frac{k}{2} + 1, 0)$ *exactly cancels* with the diagonal part. This cancellation can only occur in the half-plane $\operatorname{Re} s < -\frac{k+1}{2}$ and allows us to ignore \mathcal{R}_0^\pm as soon as it appears.

4.4. Non-Spectral Part. We conclude this section with a few remarks on the polar behavior of the non-spectral part. As it appears in $2Z_k(s + \frac{k}{2} + 1, 0)$, this term takes the form

$$\mathfrak{E}_k(s) = \frac{2\pi^k \Gamma(s+1) \zeta(s+1) \zeta(s+k)}{\Gamma(\frac{k}{2}) \Gamma(s + \frac{k}{2} + 1) \zeta^{(2)}(k)} \left(1 + \frac{1}{2^{2s+k}} - \frac{1}{2^{s+k-1}} \right). \quad (4.5)$$

This expression is analytic in the region $\operatorname{Re} s > 0$ and extends meromorphically to all of \mathbb{C} with poles $s = 0$ and $s = -1$. Potential poles at negative odd integers ≤ -3 are cancelled by trivial zeta zeros, while the existence of the poles at negative even integers depends on k .

When k is odd, $\mathfrak{E}_k(s)$ has poles at negative even integers and a double pole at $s = 1 - k$ coming from $\Gamma(s+1) \zeta(s+k)$. When k is even, zeros from $\zeta(s+1) \zeta(s+k) / \Gamma(s + \frac{k}{2} + 1)$ cancel all but $\lfloor \frac{k}{4} \rfloor$ of these additional poles, leaving only poles at 0, -1 , and each negative even integer greater than $-1 - \frac{k}{2}$.

We compute the residue at $s = -1$ to be

$$\operatorname{Res}_{s=-1} \mathfrak{E}_k(s) = -\frac{\pi^k \zeta(k-1)}{\zeta^{(2)}(k) \Gamma(\frac{k}{2})^2},$$

which perfectly cancels the corresponding pole from the diagonal part in (4.3).

5. ANALYSIS OF $D(s, P_k \times P_k)$

We now analyze $D(s, P_k \times P_k)$. Through the decomposition in (2.2), we relate $D(s, P_k \times P_k)$ to $D(s, S_k \times S_k)$, which further decomposes in terms of $W_k(s)$ from (2.5). Building on the analysis from the previous sections, we will show surprising amounts of cancellation in the poles and residues of $D(s, P_k \times P_k)$.

It is helpful to combine the two decompositions (2.2) and (2.5) into the following unified formula for $D(s, P_k \times P_k)$:

$$D(s, P_k \times P_k) = \zeta(s+k-2) + W_k(s-2) + V_k^2 \zeta(s-2) \quad (5.1)$$

$$- 2V_k \zeta(s + \frac{k}{2} - 2) - 2V_k L(s-1, \theta^k) \quad (5.2)$$

$$+ \frac{1}{2\pi i} \int_{(\sigma)} W_k(s-2-z) \zeta(z) \frac{\Gamma(z) \Gamma(s+k-2-z)}{\Gamma(s+k-2)} dz \quad (5.3)$$

$$- \frac{2V_k}{2\pi i} \int_{(\sigma)} L(s-1-z, \theta^k) \zeta(z) \frac{\Gamma(z) \Gamma(s + \frac{k}{2} - 2 - z)}{\Gamma(s + \frac{k}{2} - 2)} dz, \quad (5.4)$$

initially valid with $\operatorname{Re} s \gg 1$ and $\sigma \in (1, \operatorname{Re} s + k - 2)$.

Since the discrete part of $W_k(s-3)$ has a line of poles where $\operatorname{Re} s = \frac{3-k}{2}$, we necessarily restrict our analysis of $D(s, P_k \times P_k)$ to the half-plane $\operatorname{Re} s > \frac{3-k}{2}$. For ease of exposition, we further restrict ourselves to the half-plane $\operatorname{Re} s > 0$.

We investigate the analytic properties of $D(s, P_k \times P_k)$ by expounding each part of the decomposition given in (5.1)–(5.4). For easy reference, a

TABLE 1. Summary of Polar Data in the Half-Plane $\text{Re } s > 0$

POLE LOCATION	LINE	CONTRIBUTING TERM	RESIDUE
$s = 3$	(5.1)	$V_k^2 \zeta(s-2)$	V_k^2
$s = 3$	(5.3)	$\frac{\mathfrak{E}_k(s-3)}{s+k-3}$, from $\frac{W_k(s-3)}{s+k-3}$	V_k^2
$s = 3$	(5.4)	$-2V_k \frac{L(s-2, \theta^k)}{s+\frac{k}{2}-3}$	$-2V_k^2$
$s = 2$	(5.1)	$\mathfrak{E}_k(s-2)$, from $W_k(s-2)$	kV_k^2
$s = 2$	(5.2)	$-2V_k L(s-1, \theta^k)$	$-kV_k^2$
$s = 2$	(5.3)	$-\frac{\mathfrak{E}_k(s-2)}{2}$, from $-\frac{W_k(s-2)}{2}$	$-\frac{k}{2}V_k^2$
$s = 2$	(5.4)	$2V_k \frac{L(s-1, \theta^k)}{2}$	$\frac{k}{2}V_k^2$
$s = 3 - \frac{k}{2}$	(5.2)	$-2V_k \zeta(s + \frac{k}{2} - 2)$	$-2V_k$
$s = 3 - \frac{k}{2}$	(5.4)	$-2V_k \frac{L(s-2, \theta^k)}{s+\frac{k}{2}-3}$	$-2V_k L(1 - \frac{k}{2}, \theta^k)$
$s = 1$, if $k \neq 3$	(5.3)	$\frac{\mathfrak{E}_k(s-3)}{s+k-3}$, from $\frac{W_k(s-3)}{s+k-3}$	$\frac{\pi^k \zeta(k-2)(1+2^{3-k})}{12\Gamma(\frac{k}{2})^2 \zeta^{(2)}(k)}$
$s = 1$	(5.3)	$\frac{\mathfrak{E}_k(s-1)(s+k-2)}{12}$	$\frac{V_k^2 k(k-1)}{12}$
$s = 1$	(5.4)	$-2V_k \frac{L(s, \theta^k)(s+\frac{k}{2}-2)}{12}$	$-V_k \frac{\pi^{k/2}(\frac{k}{2}-1)}{6\Gamma(k/2)}$
$s = 4 - k$, if k odd	(5.3)	$\frac{\mathfrak{E}_k(s-3)}{s+k-3}$, from $\frac{W_k(s-3)}{s+k-3}$	double pole, see (5.7)
$s = 3 - \frac{k+1}{2}$, $k \neq 3$	(5.3)	$\frac{2\mathcal{R}_1^+(s+\frac{k}{2}-2, 0)}{s+k-3}$, from $\frac{W_k(s-3)}{s+k-3}$	$\frac{(4\pi)^{\frac{k}{2}} V(\mathcal{V}, E_0(\cdot, \frac{3}{2}))}{\pi^{3/2} \Gamma(\frac{k+1}{2})}$
$s = 3 - \frac{k+1}{2}$, $k = 3$	(5.3)	$\frac{2\mathcal{R}_1^+(s+\frac{k}{2}-2, 0)}{s+k-3}$, from $\frac{W_k(s-3)}{s+k-3}$	double pole, see (5.6)

See Proposition 2.2 for the definition of W_k , (3.15) for \mathcal{R}_1^+ , and (4.5) for \mathfrak{E}_k .

summary of the locations and residues of the poles of $D(s, P_k \times P_k)$ in the half-plane $\text{Re } s > 0$ is provided in Table 1.

Poles from terms in (5.1) and (5.2). The terms occurring in the first two lines include $W_k(s-2)$ and a collection of functions of classical interest. The poles and residues of these terms are therefore given by Theorem 4.1 or are otherwise well-known.

The $W_k(s)$ integral in (5.3). To understand the integral, we shift σ to $-3 + \epsilon$ for some small $\epsilon > 0$ and understand the resulting residues. There are residues at $z = 1$ from $\zeta(z)$, and at $z = 0$ and $z = -1$ from $\Gamma(z)$. By Cauchy's Theorem, the $W_k(s)$ integral in (5.3) is equal to

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(-3+\epsilon)} W_k(s-2-z) \zeta(z) \frac{\Gamma(z) \Gamma(s+k-2-z)}{\Gamma(s+k-2)} dz \\ & + \frac{W_k(s-3)}{s+k-3} - \frac{W_k(s-2)}{2} + \frac{W_k(s-1)(s+k-2)}{12}. \end{aligned}$$

The integrand is now analytic for $\operatorname{Re} s > -1 + \epsilon$, and the poles from the z -residues can be interpreted using Theorem 4.1.

The $L(s, \theta^k)$ integral in (5.4). As with the previous integral, we shift σ to $-3 + \epsilon$ for some small $\epsilon > 0$ and understand the resulting residues. By Cauchy's Theorem, the $L(s, \theta^k)$ integral in (5.4) is equal to

$$\begin{aligned} & \frac{-2V_k}{2\pi i} \int_{(-3+\epsilon)} L(s-1-z, \theta^k) \zeta(z) \frac{\Gamma(z)\Gamma(s+\frac{k}{2}-2-z)}{\Gamma(s+\frac{k}{2}-2)} dz \\ & - 2V_k \left(\frac{L(s-2, \theta^k)}{s+\frac{k}{2}-3} - \frac{L(s-1, \theta^k)}{2} + \frac{L(s, \theta^k)(s+\frac{k}{2}-2)}{12} \right). \end{aligned}$$

The integrand is analytic for $\operatorname{Re} s > -1 + \epsilon$. As $L(s, \theta^k)$ is analytic except for a simple pole at $s = 1$, it is easy to recognize the poles with $\operatorname{Re} s > 0$ in the expression above. Note that there is an additional pole at $s = 3 - \frac{k}{2}$ coming from the denominator of $L(s-2, \theta^k)(s+\frac{k}{2}-3)^{-1}$.

5.1. Examination of Poles and their Cancellation. We now begin a polar analysis of $D(s, P_k \times P_k)$ in the half-plane $\operatorname{Re} s > 0$. With reference to Table 1, we see at once that the residues of $D(s, P_k \times P_k)$ at $s = 3$ and $s = 2$ both vanish, hence neither of these potential poles occur.

We now address the contribution of the poles at $s = 3 - \frac{k}{2}$, which are the rightmost potential poles in the $k = 3$ case. These poles occur in the terms $-2V_k \zeta(s + \frac{k}{2} - 2)$ and $L(s-2, \theta^k)(s + \frac{k}{2} - 3)^{-1}$, and combine to give the residue

$$\operatorname{Res}_{s=3-\frac{k}{2}} \left(-2V_k \zeta(s + \frac{k}{2} - 2) + \frac{L(s-2, \theta^k)}{s + \frac{k}{2} - 3} \right) = -2V_k (1 + L(1 - \frac{k}{2}, \theta^k)).$$

We evaluate $L(1 - \frac{k}{2}, \theta^k)$ using the functional equation of $L(s, \theta^k)$,

$$\pi^{-s-\frac{k}{2}+1} \Gamma(s + \frac{k}{2} - 1) L(s, \theta^k) = \pi^{s-1} \Gamma(1-s) L(2 - \frac{k}{2} - s, \theta^k),$$

and conclude that

$$L(1 - \frac{k}{2}, \theta^k) = \frac{\Gamma(\frac{k}{2})}{\pi^{\frac{k}{2}}} \lim_{s \rightarrow 0} \frac{L(1-s, \theta^k)}{\Gamma(s)} = -\frac{\Gamma(\frac{k}{2})}{\pi^{\frac{k}{2}}} \operatorname{Res}_{s=1} L(s, \theta^k) = -1.$$

Therefore, the residue at $s = 3 - \frac{k}{2}$ is exactly 0, and so this pole also cancels.

There is a simple pole at $s = 3 - \frac{k+1}{2}$ in the case $k \geq 4$, with residue

$$\operatorname{Res}_{s=3-\frac{k+1}{2}} \frac{2\mathcal{R}_1^+(s + \frac{k}{2} - 2, 0)}{s + k - 3} = \frac{(4\pi)^{\frac{k}{2}} V}{\pi^{\frac{3}{2}} \Gamma(\frac{k+1}{2})} \langle \mathcal{V}, E_0(\cdot, \frac{3}{2}) \rangle. \quad (5.5)$$

When $k = 3$, this term is a double pole at $s = 1$, with principal part

$$-\frac{\pi^2}{3\zeta^{(2)}(3)(s-1)^2} + \frac{\pi^2(1-\gamma-\log(4\pi))}{3\zeta^{(2)}(3)(s-1)} + \frac{8Va_0}{(s-1)}, \quad (5.6)$$

in which a_0 is the constant term in the Laurent series for the meromorphic continuation of $\langle \mathcal{V}, E_0(\cdot, \frac{3}{2}) \rangle$ at $s = \frac{3}{2}$.

In general, the poles at $s = 1$ do not cancel, and constitute the leading polar term. There are always simple poles coming from $\mathfrak{E}_k(s-1)(s+k-2)/12$ and $-2V_k L(s, \theta^k)(s + \frac{k}{2} - 2)/12$, which jointly contribute the residue

$$\frac{1}{24}k^2V_k^2.$$

There is also a pole at $s = 1$ coming from $\mathfrak{E}_k(s-3)(s+k-3)^{-1}$, but the nature of this pole depends on k . There are two cases. If $k > 3$, there is a simple pole with residue

$$\frac{\pi^k \zeta(k-2)}{12 \Gamma(\frac{k}{2})^2 \zeta^{(2)}(k)} \left(1 + 2^{3-k}\right).$$

If $k = 3$, then there is a double pole with principal part

$$\frac{2\pi^2}{3\zeta^{(2)}(3)(s-1)^2} + \frac{\pi^2(2\gamma + \log 2 - 24\zeta'(-1))}{3\zeta^{(2)}(3)(s-1)}, \quad (5.7)$$

Altogether, the analysis of §5.1 leads to the following theorem.

Theorem 5.1. *The Dirichlet series $D(s, P_k \times P_k)$, defined originally in the right half-plane $\operatorname{Re} s > 3$ by the series*

$$\sum_{m=1}^{\infty} \frac{P_k(m)^2}{m^{s+k-2}},$$

has a meromorphic continuation to \mathbb{C} given by (5.1)–(5.4) and is analytic in the right half-plane $\operatorname{Re} s > 1$, with a pole at $s = 1$. In the case $k \geq 4$ this pole is simple, with residue

$$\frac{k^2}{24}V_k^2 + \frac{\pi^k \zeta(k-2)}{12 \Gamma(\frac{k}{2})^2 \zeta^{(2)}(k)} \left(1 + 2^{3-k}\right).$$

In the case $k = 3$ this is a double pole, with principal part given by

$$\frac{\pi^2}{3\zeta^{(2)}(3)(s-1)^2} + \frac{\pi^2(1 + \gamma - \log(2\pi) - 24\zeta'(-1) + 2\zeta^{(2)}(3))}{3\zeta^{(2)}(3)(s-1)} + \frac{8a_0V}{s-1}.$$

The function $D(s, P_k \times P_k)$ is otherwise analytic in the right half-plane $\operatorname{Re} s > \frac{3-k}{2}$ save for finitely many poles at non-positive integers and, for $k > 3$, an additional simple pole at $s = \frac{5-k}{2}$ with residue given by (5.5).

Remark 5.2. In the process of proving this theorem, we have also shown that $D(s, S_k \times S_k)$ has a meromorphic continuation to \mathbb{C} . The poles and residues of $D(s, S_k \times S_k)$ can be recovered from the analysis of $D(s, P_k \times P_k)$ and the decomposition (2.2).

Remark 5.3. The simple pole at $s = \frac{5-k}{2}$ is particularly interesting in the case $k = 4$, when it appears in the right half-plane $\operatorname{Re} s > 0$. In this case, Borwein and Choi [BC03] give the explicit analytic continuation

$$\sum_{m=1}^{\infty} \frac{r_4(m)^2}{m^s} = 64 \frac{(2^{6-3s} - 5 \cdot 2^{3-2s} + 2^{1-s} + 1)\zeta(s-2)\zeta^2(s-1)\zeta(s)}{(1 + 2^{1-s})\zeta(2s-2)},$$

which can be used to evaluate the inner product $\langle \mathcal{V}, E_0(\cdot, \frac{3}{2}) \rangle$ appearing in (5.5) via Zagier regularization (§4.1). We conclude that the residue of $D(s, P_4 \times P_4)$ at $s = \frac{1}{2}$ is given by

$$C'_4 := \frac{16(9\sqrt{2} - 8)\zeta(\frac{1}{2})\zeta(\frac{3}{2})^2\zeta(\frac{5}{2})}{7\pi^2\zeta(3)}.$$

6. SMOOTH SECOND MOMENT

In this section, we use the meromorphic properties of $D(s, P_k \times P_k)$ to prove our main smoothed result regarding estimates for $\sum P_k(n)^2 e^{-n/X}$. Key to this approach is the exponential cutoff transform

$$\frac{1}{2\pi i} \int_{(4)} D(s, P_k \times P_k) X^{s+k-2} \Gamma(s+k-2) ds = \sum_{n \geq 1} P_k(n)^2 e^{-n/X}. \quad (6.1)$$

We may evaluate the left-hand side of the inverse Mellin transform in (6.1) by decomposing $D(s, P_k \times P_k)$ as in (5.1)–(5.4) and then shifting the lines of integration from $\text{Re } s = 4$ to $\text{Re } s = \epsilon$. From Theorem 5.1, we understand that these integration shifts pass by a pole at $s = 1$ (which is a double pole for $k = 3$) and a pole at $s = \frac{1}{2}$ (if $k = 4$).

Provided that the integral in (6.1) converges away from poles on each abscissa $(\sigma) \in (0, 4)$, we would have

$$\begin{aligned} \sum_{n \geq 1} P_k(n)^2 e^{-n/X} &= \delta_{[k=3]} C'_3 X^{k-1} (\log X + 1 - \gamma) + C_k \Gamma(k-1) X^{k-1} \\ &+ \delta_{[k=4]} \Gamma(\frac{5}{2}) C'_4 X^{k-\frac{3}{2}} + \frac{1}{2\pi i} \int_{(\epsilon)} D(s, P_k \times P_k) X^{s+k-2} \Gamma(s+k-2) ds. \end{aligned} \quad (6.2)$$

Here, the constants C_k , C'_3 , and C'_4 are given explicitly by the Laurent coefficients of $D(s, P_k \times P_k)$ about its singular points, as described in Remark 1.2 and Theorem 5.1.

Since $\Gamma(s)$ experiences exponential decay as $|\text{Im } s| \rightarrow \infty$, it suffices to show that $D(s, P_k \times P_k)$ grows at most polynomially in $|\text{Im } s|$. We will accomplish this through a series of lemmas.

Lemma 6.1. *The function $W_k(s)$ is bounded polynomially in $|\text{Im } s|$ away from poles in vertical strips.*

Proof. We prove this by showing that the diagonal, non-spectral, discrete, and continuous parts of $W_k(s)$ grow at most polynomially in $|\text{Im } s|$.

For the diagonal part this is a consequence of the Phragmén-Lindelöf principle and the existence of a functional equation to give bounds for $L(s, \theta^k \times \theta^k)$ in a left half-plane. (See §4.1.)

For the non-spectral part $\mathfrak{E}_k(s)$, we obtain at most polynomial growth in $|\text{Im } s|$ as a consequence of polynomial bounds on $\zeta(s)$ and Stirling's approximation for the gamma ratio $\Gamma(s+1)/\Gamma(s + \frac{k}{2} + 1)$.

In the continuous part, we must address the growth of \mathcal{R}_{-j}^{\pm} as well as the integral (3.13). To bound

$$2\mathcal{R}_1^+(s + \frac{k}{2} + 1, 0) = (4\pi)^{\frac{k}{2}} V \frac{\Gamma(s + \frac{k}{2} + \frac{1}{2}) \pi^{s + \frac{k-1}{2}} \langle \mathcal{V}, E_0(\cdot, 1 - \frac{k}{2} - \bar{s}) \rangle}{\Gamma(s + \frac{k}{2} + 1) \Gamma(s + k)},$$

we recall that $\langle \mathcal{V}, E_0(\cdot, 1 - \frac{k}{2} - \bar{s}) \rangle$ may be identified with an L -function through (4.2), and therefore grows like a gamma function multiplied by an L -function of polynomial growth. Via Stirling's approximation we see that the exponential contributions within \mathcal{R}_1^{\pm} cancel, so \mathcal{R}_1^{\pm} grows at most polynomially in $|\operatorname{Im} s|$. Further terms \mathcal{R}_{-j}^{\pm} may be treated in the same way.

To complete our analysis of the continuous part of $W_k(s)$ we need only estimate (3.13) in various vertical strips. To do so, we note that $\langle \mathcal{V}, E_a(\cdot, \frac{1}{2} - \bar{z}) \rangle / \Gamma(\frac{1}{2} + z)$ and $\zeta_a(s, z)$ experience at most polynomial growth in $|\operatorname{Im} z|$ and $|\operatorname{Im} s|$, and that Stirling's approximation gives

$$\begin{aligned} G(s, z) &= \frac{\Gamma(s - \frac{1}{2} + z) \Gamma(s - \frac{1}{2} - z)}{\Gamma(s + \frac{k}{2} - 1) \Gamma(s)} \\ &\ll |\operatorname{Im}(s - z)|^{\operatorname{Re} s} |\operatorname{Im}(s + z)|^{\operatorname{Re} s} |\operatorname{Im} s|^A e^{-\pi \max(|\operatorname{Im} s|, |\operatorname{Im} z|) + \pi |\operatorname{Im} s|} \end{aligned}$$

when $\operatorname{Re} z = 0$, for some constant A .

In the z -interval of length $2|\operatorname{Im} s|^{1+\epsilon}$ where $|\operatorname{Im} z| < |\operatorname{Im} s|^{1+\epsilon}$, the exponential factors cancel and the integrand experiences polynomial growth in $|\operatorname{Im} s|$. If $|\operatorname{Im} z| > |\operatorname{Im} s|^{1+\epsilon}$, the integrand decays exponentially. In total, the integral contributes only polynomial growth.

Finally, we address the discrete part of $W_k(s)$. Here, [Kir15, Proposition 14] shows that the inner products $\langle \mathcal{V}, \mu_j \rangle$ decay exponentially in $|t_j|$; namely,

$$\sum_{T \leq |t_j| \leq 2T} |\langle \mathcal{V}, \mu_j \rangle|^2 \ll T^{4k+2} e^{-\pi T}.$$

This exponential decay is balanced by exponential growth within the Fourier coefficients $\rho_j(h)$. With the estimate

$$\sum_{T \leq |t_j| \leq 2T} |\rho_j(h)|^2 e^{-\pi T} \ll_h T^2$$

given in [HH16, (4.3)], our previous bound on $G(s, z)$, and the observation that $L(s, \mu_j)$ experiences polynomial growth in vertical strips, we bound the discrete part of $W_k(s)$ polynomially in $|\operatorname{Im} s|$ via partial summation. \square

A second lemma will be used to bound the growth of the two Mellin-Barnes integrals (5.3) and (5.4) that appear in the meromorphic continuation of $D(s, P_k \times P_k)$.

Lemma 6.2. *Let $F(s)$ be a function that grows polynomially in $|\operatorname{Im} s|$ in vertical strips and let c be fixed. There exists $M > 0$ such that*

$$\frac{1}{2\pi i} \int_{(\sigma)} F(s - z) \zeta(z) \frac{\Gamma(z) \Gamma(s + c - z)}{\Gamma(s + c)} dz \ll |\operatorname{Im} s|^M,$$

where the implicit constant does not depend on $|\operatorname{Im} s|$.

Proof. By Stirling's approximation and polynomial growth in vertical strips for both $F(s-z)$ and $\zeta(z)$, we bound our integrand by

$$|\operatorname{Im}(s-z)|^A |\operatorname{Im} z|^B |\operatorname{Im} s|^C e^{-\frac{\pi}{2}|\operatorname{Im} z| - \frac{\pi}{2}|\operatorname{Im}(s-z)| + \frac{\pi}{2}|\operatorname{Im} s|}$$

for some A, B, C independent of $|\operatorname{Im} s|$ and $|\operatorname{Im} z|$. Growth and decay of the integrand depends on the relative sizes of $\operatorname{Im} s$, $\operatorname{Im} z$, and $\operatorname{Im}(s-z)$. By casework we conclude that the integrand has exponential decay in $|\operatorname{Im} z|$ everywhere except when $|\operatorname{Im} z| \leq |\operatorname{Im} s|$, in which case the exponentials cancel. Thus the integrand is polynomially bounded and effectively supported on an interval of length $2|\operatorname{Im} s|^{1+\epsilon}$, leading to a polynomial bound in $|\operatorname{Im} s|$ overall. \square

Combining our lemmas, we bound $D(s, P_k \times P_k)$ in vertical strips and prove the following theorem.

Theorem 6.3. *For $k \geq 3$ and any $\epsilon > 0$,*

$$\begin{aligned} \sum_{n=1}^{\infty} P_k(n)^2 e^{-n/X} &= \delta_{[k=3]} C'_3 X^{k-1} (\log X + 1 - \gamma) + C_k \Gamma(k-1) X^{k-1} \\ &\quad + \delta_{[k=4]} C'_4 \Gamma(k - \tfrac{3}{2}) X^{k-\frac{3}{2}} + O_{\epsilon}(X^{k-2+\epsilon}), \end{aligned}$$

where C_k , C'_3 , and C'_4 are the explicit constants described in Remark 1.2.

Proof. As described at the start of this section, it suffices to shift the line of integration as in (6.2). To justify this contour shift, we bound $D(s, P_k \times P_k)$ polynomially in $|\operatorname{Im} s|$ in vertical strips. We do so by showing a contribution of at most polynomial growth for each term in (5.1)–(5.4).

In (5.1) these bounds follow from Lemma 6.1 and polynomial estimates for the Riemann zeta function in vertical strips. For (5.2) we require a polynomial bound on $L(s, \theta^k)$ in vertical strips as well, which follows from the functional equation of $L(s, \theta^k)$ and the Phragmén-Lindelöf Principle. Finally, since $W_k(s)$ and $L(s, \theta^k)$ experience polynomial growth in vertical strips, Lemma 6.2 gives a polynomial bound in $|\operatorname{Im} s|$ in (5.3) and (5.4). \square

Remark 6.4. The leading constants C'_3 and C_k ($k \geq 4$) are described explicitly in Remark 1.2. In particular, we may verify that they are positive.

For small $k > 3$ it is not difficult to list the precise locations of the poles of $D(s, P_k \times P_k)$ in the right half-plane $\operatorname{Re} s > \frac{3-k}{2}$ and derive additional main terms and improved error estimates in Theorem 6.3. For example, there exist constants D_4 and D_5 for which

$$\begin{aligned} \sum_{n \geq 1} P_4(n)^2 e^{-n/X} &= 2C_4 X^3 + C'_4 \Gamma(\tfrac{5}{2}) X^{\frac{5}{2}} + D_4 X^2 + O(X^{\frac{3}{2}+\epsilon}), \\ \sum_{n \geq 1} P_5(n)^2 e^{-n/X} &= 6C_5 X^4 + D_5 X^3 + O(X^{2+\epsilon}). \end{aligned}$$

The existence of infinitely many poles for $D(s, P_k \times P_k)$ on the line $\operatorname{Re} s = \frac{3-k}{2}$ suggests that these are essentially the best smooth results possible.

7. SHARP SECOND MOMENT

To produce a second moment result without smoothing, we introduce two additional smooth cutoff transforms and study them in tandem. The first of these, which we call $v_y(x)$, is a generic smooth cutoff coming from a Mellin transform of a function with compact support, similar to the cutoffs the authors used in [HKLDW17c].

Proposition 7.1. *Let $v_y(x)$ be a function satisfying*

- a. $v_y(x) = 1$ for $x \in [0, 1]$, and $v_y(x) \leq 1$ for all x ,
- b. $v_y(x)$ is supported on $[0, 1 + 1/y]$,
- c. $v_y(x)$ is smooth.

Let $V_y(s)$ be the Mellin transform of $v_y(x)$. Then the following are true:

- 1. $V_y(s) = 1/s + O_s(1/y)$,
- 2. $V_y'(s) = -1/s^2 + O_s(1/y)$,
- 3. For integer $m \geq 1$, we have $V_y(s) \ll \frac{1}{y} \left(\frac{y}{1+|\operatorname{Im} s|} \right)^m$.

Proof. The first statement follows from the definition of $v_y(x)$, as

$$V_y(s) = \int_0^\infty v_y(t) t^s \frac{dt}{t} = \int_0^1 t^{s-1} dt + \int_1^{1+1/y} v_y(t) t^s \frac{dt}{t} = \frac{1}{s} + O_s\left(\frac{1}{y}\right).$$

For (2), differentiate under the integral sign to obtain

$$V_y'(s) = \int_0^1 t^s \log t \frac{dt}{t} + \int_1^{1+1/y} v_y(t) t^s \log t \frac{dt}{t} = -\frac{1}{s^2} + O_s\left(\frac{1}{y}\right).$$

Repeated integration by parts on the definition of $V_y(s)$ proves (3). \square

We will also use an integral transform that concentrates in the region where $v_y(x)$ decreases to 0, namely

$$\frac{1}{i} \int_{(\sigma)} X^s \exp\left(\frac{\pi s^2}{y^2}\right) \frac{ds}{y} = \exp\left(-\frac{y^2 \log^2 X}{4\pi}\right). \quad (7.1)$$

Note that this transform experiences exponential decay outside of the range $[1 - 1/y, 1 + 1/y]$. (See [HKLDW17b, §4] for more on this integral transform and its properties.)

Combining these two smooth integral transforms allows one to produce sharp estimates for $P_k(n)^2$, and in general, for sums of non-negative terms.

Proposition 7.2. *For $k \geq 3$ we have*

$$\begin{aligned} \sum_{n \leq X} P_k(n)^2 &= \frac{1}{2\pi i} \int_{(k)} D(s - k + 2, P_k \times P_k) X^s V_y(s) ds \\ &\quad + O\left(\frac{1}{2\pi i} \int_{(2)} D(s - k + 2, P_k \times P_k) X^s \exp\left(\frac{\pi s^2}{y^2}\right) \frac{ds}{y}\right). \end{aligned}$$

Proof. With $\sigma \gg 1$ so that the integral converges absolutely, the definitions of $v_y(x)$ and $V_y(s)$ give

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(\sigma)} D(s-k+2, P_k \times P_k) X^s V_y(s) ds \\ &= \frac{1}{2\pi i} \sum_{n \geq 1} P_k(n)^2 \int_{(\sigma)} \left(\frac{X}{n}\right)^s V_y(s) ds = \sum_{n=1}^{\infty} P_k(n)^2 v_y\left(\frac{n}{X}\right) \\ &= \sum_{n \leq X} P_k(n)^2 + \sum_{X < n \leq X+X/y} P_k(n)^2 v_y\left(\frac{n}{X}\right). \end{aligned}$$

For all $n \in [X, X + X/y]$, we have the uniform lower bound

$$\exp\left(-\frac{y^2 \log^2(X/n)}{4\pi}\right) \gg \exp\left(-\frac{y^2 \log^2(1+1/y)}{4\pi}\right) \gg 1.$$

Therefore, as $P_k(n)^2 \geq 0$, we have

$$\sum_{X < n \leq X+X/y} P_k(n)^2 v_y\left(\frac{n}{X}\right) \ll \sum_{|n-X| \leq X/y} P_k(n)^2 \exp\left(-\frac{y^2 \log^2(X/n)}{4\pi}\right).$$

We recognize this last sum as the transform (7.1) after extending the range of the sum to all $n \geq 1$. \square

Therefore, to understand $\sum_{n \leq X} P_k(n)^2$, it suffices to understand the two integral transforms in Proposition 7.2. We begin with $v_y(x)$, the smooth cutoff of compact support.

Lemma 7.3. *There exists $M > 0$ such that*

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(k)} D(s-k+2, P_k \times P_k) X^s V_y(s) ds \\ &= \delta_{[k=3]} \frac{C'_3 X^2}{2} (\log X - \tfrac{1}{2}) + \frac{C_k X^{k-1}}{k-1} + O\left(\frac{X^{k-1} \log X}{y} + X^{k-\frac{5}{4}} y^M\right). \end{aligned}$$

Proof. Shifting the contour left to $\operatorname{Re} s = (k-1-\frac{1}{4})$ passes a pole at $s = k-1$. By Theorem 5.1 and Proposition 7.1, for $k > 3$ this pole is simple and has residue

$$C_k X^{k-1} V_y(k-1) = \frac{C_k}{k-1} X^{k-1} + O\left(\frac{X^{k-1}}{y}\right).$$

When $k = 3$, the pole is a double pole and contributes the residue

$$\begin{aligned} & C'_3 V_y(2) X^2 \log X + (C_3 V_y(2) + C'_3 V'_y(2)) X^2 \\ &= \frac{C'_3}{2} X^2 \log X + \left(\frac{C_3}{2} - \frac{C'_3}{4}\right) X^2 + O\left(\frac{X^2 \log X}{y}\right). \end{aligned}$$

By Lemmas 6.1 and 6.2, there is an $M > 0$ such that $D(s, P_k \times P_k) \ll |\operatorname{Im} s|^{M-1}$ on the line $\operatorname{Re} s = (k-1-\frac{1}{4})$. Bounding the shifted integral by

$$\frac{X^{k-1-\frac{1}{4}}}{y} \int_{-\infty}^{\infty} (1+|t|)^{M-1} \left(\frac{y}{1+|t|} \right)^{M+1} dt = O\left(X^{k-1-\frac{1}{4}} y^M\right)$$

completes the proof. \square

A similar argument bounds the size of the concentrating integral transform, and correspondingly, the size of the error term in Proposition 7.2.

Lemma 7.4. *There exists $M > 0$ such that*

$$\frac{1}{2\pi i} \int_{(k)} D(s-k+2, P_k \times P_k) X^s \exp\left(\frac{\pi s^2}{y^2}\right) \frac{ds}{y} \ll \frac{X^{k-1} \log X}{y} + X^{k-\frac{5}{4}} y^M.$$

Proof. Shift the contour left to $(k-1-\frac{1}{4})$. This passes a pole at $s = k-1$ with residue bounded by

$$O\left(\frac{X^{k-1} \log X}{y}\right).$$

There exists an M such that $D(s, P_k \times P_k) \ll |\operatorname{Im} s|^M$ when $\operatorname{Re} s = (k-1-\frac{1}{4})$, which bounds the shifted integral by

$$\frac{X^{k-1-\frac{1}{4}}}{y} \int_{-\infty}^{\infty} (1+|t|)^M \exp\left(-\frac{\pi t^2}{y^2}\right) dt \ll X^{k-1-\frac{1}{4}} y^M,$$

as claimed. \square

Combining Proposition 7.2, Lemma 7.3, and Lemma 7.4 gives the following theorem.

Theorem 7.5. *For each $k \geq 3$ there exists a $\lambda > 0$ such that*

$$\sum_{n \leq X} P_k(n)^2 = \delta_{[k=3]} X^{k-1} \left(\frac{C'_3}{2} \log X - \frac{C'_3}{4} \right) + \frac{C_k}{k-1} X^{k-1} + O_\lambda(X^{k-1-\lambda}).$$

The constants C'_3 and C_k are the same constants as in Remark 1.2.

Proof. Without loss of generality, suppose $M \geq 1$. Take $y = X^{\frac{1}{8M}}$ in the definition of $v_y(x)$. Then for any $\epsilon > 0$, one can take $\lambda = \frac{1}{8M} - \epsilon$. \square

8. LAPLACE TRANSFORM

Theorem 6.3 may be considered as a discrete Laplace transform of the mean square of the lattice point discrepancy. Building upon this result, one can obtain asymptotics for the continuous Laplace transform

$$\int_0^\infty P_k(t)^2 e^{-t/X} dt. \quad (8.1)$$

In this section, we prove the following estimate for the continuous Laplace transform of $P_k(t)^2$.

Theorem 8.1. *The Laplace transform of the second moment of the lattice point discrepancy in dimensions $k \geq 3$ satisfies*

$$\begin{aligned} \int_0^\infty P_k(t)^2 e^{-t/X} dt &= \delta_{[k=3]} C'_3 X^{k-1} (\log X + 1 - \gamma) + \delta_{[k=4]} C'_4 \Gamma(k - \tfrac{3}{2}) X^{k-\frac{3}{2}} \\ &\quad + C_k \Gamma(k-1) X^{k-1} - \frac{\Gamma(k-1) \pi^k}{6 \Gamma(\frac{k}{2})^2} X^{k-1} + O(X^{k-2+\epsilon}), \end{aligned}$$

where the constants are the same constants as in Remark 1.2.

Remark 8.2. It is possible to adapt the method of the proof of Theorem 8.1 to obtain further secondary terms and decrease the error to $O(X^{\frac{k-1}{2}+\epsilon})$.

Our proof of Theorem 8.1 begins with the identity

$$P_k(t) = S_k(t) - V_k t^{\frac{k}{2}} = S_k(\lfloor t \rfloor) - V_k t^{\frac{k}{2}} = P_k(\lfloor t \rfloor) + V_k \lfloor t \rfloor^{\frac{k}{2}} - V_k t^{\frac{k}{2}}.$$

It follows that

$$P_k(t)^2 = P_k(\lfloor t \rfloor)^2 + V_k^2 (\lfloor t \rfloor^{\frac{k}{2}} - t^{\frac{k}{2}})^2 + 2V_k P_k(\lfloor t \rfloor) (\lfloor t \rfloor^{\frac{k}{2}} - t^{\frac{k}{2}}). \quad (8.2)$$

We will compute the Laplace transform (8.1) by computing it separately for each term in (8.2). We begin with the first term in (8.2), which is very nearly equivalent to the sum studied in Theorem 6.3.

Lemma 8.3 (First term in the Laplace transform of (8.2)). *We have*

$$\begin{aligned} \int_0^\infty P_k(\lfloor t \rfloor)^2 e^{-t/X} dt &= \delta_{[k=3]} C'_3 X^{k-1} (\log X + 1 - \gamma) + C_k \Gamma(k-1) X^{k-1} \\ &\quad + \delta_{[k=4]} C'_4 \Gamma(k - \tfrac{3}{2}) X^{k-\frac{3}{2}} + O_\epsilon(X^{k-2+\epsilon}), \end{aligned}$$

Proof. We directly compute

$$\begin{aligned} \int_0^\infty P_k(\lfloor t \rfloor)^2 e^{-t/X} dt &= \sum_{n \geq 0} P_k(n)^2 \int_n^{n+1} e^{-t/X} dt \\ &= X(1 - e^{-1/X}) \sum_{n \geq 0} P_k(n)^2 e^{-n/X}. \end{aligned}$$

The sum is essentially the object of study in Theorem 6.3. Noting that $X(1 - e^{-1/X}) = 1 + O(1/X)$ and simplifying completes the proof. \square

The second term in (8.2) can be understood through Abel summation.

Lemma 8.4 (Second term in the Laplace transform of (8.2)). *We have*

$$V_k^2 \int_0^\infty (\lfloor t \rfloor^{\frac{k}{2}} - t^{\frac{k}{2}})^2 e^{-t/X} dt = \frac{k^2 V_k^2 \Gamma(k-1)}{12} X^{k-1} + O(X^{k-2+\epsilon}).$$

Proof. Split the integral and distribute the square to obtain

$$V_k^2 \int_0^\infty (\lfloor t \rfloor^{\frac{k}{2}} - t^{\frac{k}{2}})^2 e^{-t/X} dt = V_k^2 \sum_{n \geq 0} \int_n^{n+1} (n^k + t^k - 2n^{\frac{k}{2}} t^{\frac{k}{2}}) e^{-t/X} dt.$$

We break this into three parts, which we call I_1, I_2 , and I_3 , corresponding to the three terms in the parenthetical.

The first of these is

$$\begin{aligned} I_1 &= V_k^2 \sum_{n \geq 0} \int_n^{n+1} n^k e^{-t/X} dt = V_k^2 X (1 - e^{-1/X}) \sum_{n \geq 0} n^k e^{-n/X} \\ &= \frac{V_k^2 X (1 - e^{-1/X})}{2\pi i} \int_{(\sigma)} \zeta(s - k) X^s \Gamma(s) ds, \end{aligned}$$

valid for $\sigma > k + 1$. The gamma function gives exponential decay in vertical strips, which justifies shifting the line of integration to $\sigma = 1$. This passes a pole from the zeta function at $s = k + 1$ and shows that

$$I_1 = V_k^2 \Gamma(k + 1) X^{k+1} (1 - e^{-1/X}) + O(X).$$

The second part, I_2 , is essentially the Laplace transform of t^k . Thus

$$I_2 = V_k^2 \int_0^\infty t^k e^{-t/X} dt = V_k^2 \Gamma(k + 1) X^{k+1}.$$

It remains to estimate the third part, which takes the form

$$I_3 = -2V_k^2 \sum_{n \geq 0} n^{\frac{k}{2}} \int_n^{n+1} t^{\frac{k}{2}} e^{-t/X} dt.$$

By Abel summation, this can be written

$$I_3 = 2V_k^2 \int_0^\infty \left(\sum_{n \leq t} n^{\frac{k}{2}} \right) \left((t+1)^{\frac{k}{2}} e^{-(t+1)/X} - t^{\frac{k}{2}} e^{-t/X} \right) dt.$$

To understand the partial sum over $n \leq t$, we apply the Perron formula

$$\sum_{n \leq t} n^{\frac{k}{2}} = \frac{1}{2\pi i} \int_{(\sigma)} \zeta(s - \frac{k}{2}) t^s \frac{ds}{s},$$

valid for $\sigma > \frac{k}{2} + 1$ and $t \notin \mathbb{Z}$ in the sense of Cauchy's principal value. Inserting Perron's formula and swapping the order of integration shows that

$$I_3 = \frac{2V_k^2}{2\pi i} \int_{(\sigma)} \zeta(s - \frac{k}{2}) \int_0^\infty \left((t+1)^{\frac{k}{2}} t^s e^{-(t+1)/X} - t^{s+\frac{k}{2}} e^{-t/X} \right) dt \frac{ds}{s}. \quad (8.3)$$

We now evaluate the t -integral in (8.3) directly. The second term in the t -integral is essentially the definition of the gamma function, and gives $-X^{s+\frac{k}{2}+1} \Gamma(s + \frac{k}{2} + 1)$. To evaluate the first half of the t -integral, we first represent the exponential as an inverse Mellin transform,

$$\int_0^\infty (t+1)^{\frac{k}{2}} t^s e^{-(t+1)/X} dt = \int_0^\infty \frac{(t+1)^{\frac{k}{2}} t^s}{2\pi i} \int_{(\sigma')} \left(\frac{X}{t+1} \right)^u \Gamma(u) du dt, \quad (8.4)$$

valid for $\sigma' > 0$. Swapping the order of integration and recalling an integral representation of the beta function,

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^\infty \frac{t^\alpha}{(1+t)^{\alpha+\beta}} \frac{dt}{t},$$

we recognize the order-swapped t -integral from (8.4) as $B(s+1, u - \frac{k}{2} - s - 1)$, provided that $\sigma' > \operatorname{Re} s + \frac{k}{2} + 1$.

Combining these two evaluations together, we see that the t -integral in (8.3) can be written as

$$f(s) = \frac{\Gamma(s+1)}{2\pi i} \int_{(\sigma')} \frac{\Gamma(u)\Gamma(u - \frac{k}{2} - s - 1)}{\Gamma(u - \frac{k}{2})} X^u du - X^{s+\frac{k}{2}+1} \Gamma(s+\frac{k}{2}+1), \quad (8.5)$$

in which $\sigma' > \operatorname{Re} s + \frac{k}{2} + 1$ initially.

Inserting this expression into (8.3) shows that the integrand in I_3 is analytic in s for $\operatorname{Re} s > \frac{k}{2} + 1$. Moreover, explicit decay in $f(s)$ shows that the s -integral need not be considered as a principal value. We shift the line of s -integration to $\sigma = -1 + \epsilon$ for a small $\epsilon > 0$, which passes two poles:

(a) a pole at $s = \frac{k}{2} + 1$ from the zeta function with residue

$$\frac{2V_k^2}{\frac{k}{2} + 1} \left(\frac{\Gamma(\frac{k}{2} + 2)}{2\pi i} \int_{(\sigma')} \frac{\Gamma(u)\Gamma(u - k - 2)}{\Gamma(u - \frac{k}{2})} X^u du - X^{k+2} \Gamma(k+2) \right),$$

(b) a pole at $s = 0$ from $1/s$ with residue

$$2V_k^2 \zeta(-\frac{k}{2}) \left(\frac{1}{2\pi i} \int_{(\sigma')} \frac{\Gamma(u)}{u - \frac{k}{2} - 1} X^u du - X^{\frac{k}{2}+1} \Gamma(\frac{k}{2} + 1) \right).$$

Note that the rightmost residue of the u -integral in (8.5) exactly cancels the second term in (8.5). Thus (8.5) may be represented by a single contour integral over $\sigma' \in (\operatorname{Re} s + \frac{k}{2}, \operatorname{Re} s + \frac{k}{2} + 1)$, and is therefore $O(X^{\operatorname{Re} s + \frac{k}{2} + \epsilon})$. The shifted integral (8.3), taken with $\sigma = -1 + \epsilon$, is then $O(X^{k-2})$, so it remains only to understand the residues (a) and (b).

In the residue at $s = \frac{k}{2} + 1$, shifting the line of u -integration to $\sigma' = k - 2 + \epsilon$ passes poles at $u = k + 2$, $k + 1$, k , and $k - 1$. Computing these residues and bounding the shifted integral shows that

$$\begin{aligned} \operatorname{Res}_{s=\frac{k}{2}+1} &= -2V_k^2 \Gamma(k+1) X^{k+1} + \frac{V_k^2 \Gamma(\frac{k}{2} + 1) \Gamma(k)}{\Gamma(\frac{k}{2})} X^k \\ &\quad - \frac{V_k^2 \Gamma(\frac{k}{2} + 1) \Gamma(k-1)}{3\Gamma(\frac{k}{2} - 1)} X^{k-1} + O(X^{k-2+\epsilon}). \end{aligned}$$

For the residue at $s = 0$, shifting the line of integration to $\sigma' = \epsilon$ extracts a residue that cancels $X^{\frac{k}{2}+1}\Gamma(\frac{k}{2}+1)$. Thus the term in (b) is $O(X^\epsilon)$, hence

$$I_3 = \operatorname{Res}_{s=\frac{k}{2}+1} + \operatorname{Res}_{s=0} + O(X^{k-2}) = -2V_k^2\Gamma(k+1)X^{k+1} + \frac{V_k^2\Gamma(k+1)}{2}X^k \\ - \frac{V_k^2\Gamma(\frac{k}{2}+1)\Gamma(k-1)}{3\Gamma(\frac{k}{2}-1)}X^{k-1} + O(X^{k-2+\epsilon}).$$

Lemma 8.4 follows by combining the estimates for I_1 , I_2 , and I_3 . To do so, we require the extended asymptotic

$$X(1 - e^{-1/X}) = 1 - \frac{1}{2X} + \frac{1}{6X^2} + O(X^{-3}).$$

Simplification completes the proof. \square

Finally, we address the last term in (8.2).

Lemma 8.5 (Third term in the Laplace transform of (8.2)). *We have*

$$2V_k \int_0^\infty P_k([t])([t]^{\frac{k}{2}} - t^{\frac{k}{2}})e^{-t/X} dt = -\frac{\pi^{\frac{k}{2}}k\Gamma(k-1)V_k}{4\Gamma(\frac{k}{2})}X^{k-1} + O(X^{k-2+\epsilon}).$$

Proof. Splitting the limits of integration gives

$$2V_k \int_0^\infty P_k([t])([t]^{\frac{k}{2}} - t^{\frac{k}{2}})e^{-t/X} dt = 2V_k \sum_{n \geq 0} P_k(n) \int_n^{n+1} (n^{\frac{k}{2}} - t^{\frac{k}{2}})e^{-t/X} dt.$$

We consider the contributions of $n^{\frac{k}{2}}$ and $t^{\frac{k}{2}}$ separately, and denote them by J_1 and J_2 , respectively. Directly computing the integrals in J_1 gives

$$J_1 = 2V_k X(1 - e^{-1/X}) \sum_{n=0}^\infty P_k(n) n^{\frac{k}{2}} e^{-n/X} \\ = (1 - e^{-1/X}) \frac{2V_k X}{2\pi i} \int_{(\sigma)} D(s - \frac{k}{2}, P_k) \Gamma(s) X^s ds, \quad (8.6)$$

in which $D(s, P_k) := \sum_{n \geq 1} P_k(n) n^{-s}$ denotes the non-normalized Dirichlet series associated to P_k . By modifying the analysis of $\sum S_k(n) n^{-(s+\frac{k}{2}-2)}$ from (2.3) and recalling that $P_k(n) = S_k(n) - V_k n^{\frac{k}{2}}$, we see that

$$D(s, P_k) = \zeta(s) + L(s - \frac{k}{2} + 1, \theta^k) - V_k \zeta(s - \frac{k}{2}) \\ + \frac{1}{2\pi i} \int_{(\sigma)} L(s - \frac{k}{2} + 1 - z, \theta^k) \zeta(z) \frac{\Gamma(z)\Gamma(s-z)}{\Gamma(s)} dz, \quad (8.7)$$

in which $L(s, \theta^k)$ is defined as in Proposition 2.1.

The function $D(s, P_k)$ admits potential poles at $s = \frac{k}{2} + 1$ (coming from a zeta function and the Mellin-Barnes integral, visible after shifting the line of integration past the pole at $z = 1$), at $s = \frac{k}{2}$ (coming from $L(s - \frac{k}{2} + 1, \theta^k)$

and the Mellin-Barnes integral), and at $s = \frac{k}{2} - 1$ (coming from the Mellin-Barnes integral), with no other poles for $\operatorname{Re} s > \frac{k}{2} - 2$. The potential pole at $s = \frac{k}{2} + 1$ cancels, while the poles at $s = \frac{k}{2}$ and $s = \frac{k}{2} - 1$ have residues

$$\operatorname{Res}_{s=\frac{k}{2}} = \frac{\pi^{\frac{k}{2}}}{2\Gamma(\frac{k}{2})} \quad \text{and} \quad \operatorname{Res}_{s=\frac{k}{2}-1} = \frac{\pi^{\frac{k}{2}}}{12\Gamma(\frac{k}{2}-1)}.$$

The integrand in (8.6) has exponential decay in vertical strips from the gamma function. Shifting the line of integration in (8.6) to $k - 2 + \epsilon$ for a small $\epsilon > 0$ shows that

$$J_1 = X(1 - e^{-1/X})V_k \left(\frac{\pi^{\frac{k}{2}}\Gamma(k)}{\Gamma(\frac{k}{2})}X^k + \frac{\pi^{\frac{k}{2}}\Gamma(k-1)}{6\Gamma(\frac{k}{2}-1)}X^{k-1} + O(X^{k-2+\epsilon}) \right).$$

Our approach to J_2 mirrors the treatment of I_2 in the proof of Lemma 8.4. Abel summation, Perron's formula, and the integral evaluation (8.5) give

$$J_2 := -2V_k \sum_{n \geq 0} P_k(n) \int_n^{n+1} t^{\frac{k}{2}} e^{-t/X} dt = \frac{2V_k}{2\pi i} \int_{(\sigma)} D(s, P_k) f(s) \frac{ds}{s},$$

where σ is sufficiently large. Shifting the line of integration to $\sigma = \frac{k}{2} - 2 + \epsilon$ for a small $\epsilon > 0$ passes poles at $s = \frac{k}{2}$ and $s = \frac{k}{2} - 1$. The analysis of these contributions is similar to the analysis of I_2 in Lemma 8.4, so we omit details. The contribution from J_2 is given by

$$J_2 = -\frac{\pi^{\frac{k}{2}}\Gamma(k)V_k}{\Gamma(\frac{k}{2})}X^k + \frac{\pi^{\frac{k}{2}}\Gamma(k-1)V_k}{3\Gamma(\frac{k}{2}-1)}X^{k-1} + O(X^{k-2+\epsilon}).$$

Adding J_1 and J_2 together completes the proof. \square

Our proof of Theorem 8.1 now follows from the three-term decomposition of $P_k(t)^2$ given in (8.2) and Lemmas 8.3, 8.4, and 8.5.

9. IMPROVING JARNIK'S INTEGRATED MEAN SQUARE ESTIMATE

As our second application of the main results of this paper, we translate Theorem 7.5 into the same language as the mean square estimate for the lattice point discrepancy on the sphere. We recall that Jarnik [Jar40] showed that

$$\int_0^X (P_3(t))^2 dt = \frac{C'_3}{2} X^2 \log X + O\left(X^2 (\log X)^{\frac{1}{2}}\right),$$

and note that the leading constant agrees with the constant in Theorem 7.5.

We will prove the following refinement of Jarnik's mean square estimate as a corollary to Theorem 7.5.

Theorem 9.1. *There exists $\lambda > 0$ such that*

$$\int_0^X P_3(t)^2 dt = \frac{C'_3}{2} X^2 \log X + \left(\frac{C_3}{2} - \frac{C'_3}{4} - \frac{\pi^2}{3} \right) X^2 + O_\lambda \left(X^{2-\lambda} \right),$$

where C'_3 and C_3 are the same constants as in Remark 1.2.

Proof. It suffices to prove Theorem 9.1 for integer X as a consequence of Heath-Brown's estimate $P_3(n) = O(n^{21/32+\epsilon})$ [HB99]. Indeed, the contribution of the integral of $(P_3(x))^2$ over $[X, X+1]$ is $O(X^{21/16+\epsilon})$, which is sufficiently small.

We rewrite Theorem 7.5 in the form

$$\int_0^X P_3(\lfloor t \rfloor)^2 dt = \frac{C'_3}{2} X^2 \log X + \left(\frac{C_3}{2} - \frac{C'_3}{4} \right) X^2 + O_\lambda(X^{2-\lambda}). \quad (9.1)$$

As a special case of (8.2) we have

$$P_3(t)^2 - P_3(\lfloor t \rfloor)^2 = 2V_3 P_3(\lfloor t \rfloor) (\lfloor t \rfloor^{\frac{3}{2}} - t^{\frac{3}{2}}) + V_3^2 (\lfloor t \rfloor^{\frac{3}{2}} - t^{\frac{3}{2}})^2.$$

The difference between (9.1) and $\int_0^X P_3(t)^2 dt$ can therefore be written as

$$2V_3 \int_0^X P_3(\lfloor t \rfloor) (\lfloor t \rfloor^{\frac{3}{2}} - t^{\frac{3}{2}}) dt + V_3^2 \int_0^X (\lfloor t \rfloor^{\frac{3}{2}} - t^{\frac{3}{2}})^2 dt. \quad (9.2)$$

The second integral in (9.2) admits the approximation

$$V_3^2 \sum_{n=0}^{X-1} \int_n^{n+1} (n^{\frac{3}{2}} - t^{\frac{3}{2}})^2 dt = V_3^2 \sum_{n=0}^{X-1} \left(\frac{3n}{4} + O(1) \right),$$

obtained by integrating each summand and then performing a series expansion in n term-by-term. Summing over $n \leq X-1$, we see that

$$V_3^2 \sum_{n=0}^{X-1} \int_n^{n+1} (n^{\frac{3}{2}} - t^{\frac{3}{2}})^2 dt = \frac{3V_3^2}{8} X^2 + O(X).$$

Now consider the first integral in (9.2). The contribution of the integral over the range $[0, 1]$ is $O(1)$. For the rest, we again break up the integral at discontinuities and integrate termwise to obtain

$$2V_3 \sum_{n=1}^{X-1} P_3(n) \int_n^{n+1} (n^{\frac{3}{2}} - t^{\frac{3}{2}}) dt = -2V_3 \sum_{n=1}^{X-1} P_3(n) \left(\frac{3\sqrt{n}}{4} + O\left(\frac{1}{n^{\frac{1}{2}}}\right) \right).$$

By Heath-Brown's bound $P_3(n) \ll n^{\frac{21}{32}+\epsilon}$, we estimate the contribution of the error term in the series expansion above by $O(X^{\frac{1}{2}+\frac{21}{32}+\epsilon})$.

Rearranging, we write the difference between $\int_0^X P_3(t)^2 dt$ and (9.1) as

$$\int_0^X P_3(t)^2 dt - \sum_{n \leq X} P_3(n)^2 = \frac{3V_3^2 X^2}{8} - \frac{3V_3}{2} \sum_{n=1}^{X-1} P_3(n) n^{\frac{1}{2}} + O(X^{\frac{37}{32}+\epsilon}). \quad (9.3)$$

It remains to estimate the partial sum $\sum_{n \leq X} P_3(n) \sqrt{n}$.

To estimate this series we use $v_y(x)$, the smooth cutoff function of compact support introduced in Proposition 7.1. Recalling the definition of the

Dirichlet series $D(s, P_3) = \sum_{n=1}^{\infty} P_3(n)/n^s$ from Lemma 8.5, we have

$$\begin{aligned} \sum_{n \leq X} P_3(n) \sqrt{n} &= \frac{1}{2\pi i} \int_{(\sigma)} D(s - \tfrac{1}{2}, P_3) X^s V_y(s) ds \\ &\quad + O\left(\sum_{X < n \leq X + X/y} |P_3(n)| n^{\frac{1}{2}} v_y\left(\frac{n}{X}\right) \right) \end{aligned} \quad (9.4)$$

for $\sigma \gg 1$. The error term is $O(X^{2+\frac{5}{32}/y})$, again by Heath-Brown's bound.

Returning to the integral in (9.4), fix a small $\epsilon > 0$ and shift the line of integration to $\operatorname{Re} s = (1 + 2\epsilon)$.

Following the decomposition in (8.7), we know that this passes simple poles at $s = 2$ and $s = \frac{3}{2}$, contributing residues

$$\frac{\pi^{\frac{3}{2}} V_y(2)}{2\Gamma(\frac{3}{2})} X^2 + O\left(X^{\frac{3}{2}}\right) = \frac{\pi}{2} X^2 + O\left(\frac{X^2}{y} + X^{\frac{3}{2}}\right).$$

Note that we have used that $V_y(s) = \frac{1}{s} + O_s(\frac{1}{y})$ from Proposition 7.1.

It remains to estimate the integral in (9.4) after shifting to $\operatorname{Re} s = 1 + 2\epsilon$, which we do by estimating each term in the decomposition (8.7) for $D(s, P_3)$.

First, we estimate the integral

$$\frac{1}{2\pi i} \int_{(-1+\epsilon)} L(s-1-z, \theta^3) \zeta(z) \frac{\Gamma(z) \Gamma(s - \frac{1}{2} - z)}{\Gamma(s - \frac{1}{2})} dz.$$

Note that $L(s, \theta^3)$ is uniformly bounded in its convergent half-plane. By the functional equation for $\zeta(z)$ and Stirling's approximation, we estimate the integrand to be bounded by

$$(1 + |s|)^{-2\epsilon} (1 + |s - z|)^{1+\epsilon} e^{-\frac{\pi}{2}(|z| + |s-z| - |s|)}.$$

When $|z| < |s|$, there is no exponential contribution and the integrand is bounded by $(1 + |s|)^{1-\epsilon}$ on an interval of length $O(|s|)$. When $|z| > |s|$, there is exponential decay in the integrand and so the contribution to the integral from this domain is $O((1 + |s|)^{1-\epsilon})$. Therefore

$$\frac{1}{2\pi i} \int_{(-1+\epsilon)} L(s-1-z, \theta^3) \zeta(z) \frac{\Gamma(z) \Gamma(s - \frac{1}{2} - z)}{\Gamma(s - \frac{1}{2})} dz \ll_{\epsilon} (1 + |s|)^{2-\epsilon}.$$

This, coupled with the Phragmén-Lindelöf convexity estimates

$$\begin{aligned} \zeta(\tfrac{1}{2} + 2\epsilon + it) &\ll (1 + |t|)^{\frac{1}{4}}, \quad \zeta(-1 + 2\epsilon + it) \ll (1 + |t|)^{\frac{3}{2}}, \\ L(2\epsilon + it, \theta^3) &\ll (1 + |t|)^1, \quad L(-1 + 2\epsilon + it, \theta^3) \ll (1 + |t|)^{\frac{5}{2}}, \end{aligned}$$

implies that $D(s - \frac{1}{2}, P_3) \ll (1 + |s|)^{2-\epsilon}$ on the line $\operatorname{Re} s = 1 + 2\epsilon$. Thus

$$\frac{1}{2\pi i} \int_{(1+2\epsilon)} D(s - \tfrac{1}{2}, P_3) X^s V_y(s) ds \ll \frac{X^{1+2\epsilon}}{y} \int_{-\infty}^{\infty} (1 + |t|)^{2-\epsilon} \left(\frac{y}{1 + |t|}\right)^3 dt$$

by Property 3 from Proposition 7.1. Referring back to (9.4), we conclude that

$$\sum_{n \leq X} P_3(n) n^{\frac{1}{2}} = \frac{\pi}{2} X^2 + O\left(\frac{X^{2+\frac{5}{32}}}{y} + X^{\frac{3}{2}} + X^{1+2\epsilon} y^2\right).$$

We minimize the error in this estimate by taking $y = X^\alpha$ with $\alpha = \frac{37}{96} - \frac{2\epsilon}{3}$, and conclude via (9.3) that

$$\int_0^X P_3(t)^2 dt - \sum_{n \leq X} P_3(n)^2 = -\frac{\pi^2}{3} X^2 + O\left(X^{2-\frac{11}{48}+\epsilon}\right).$$

Our theorem now follows as a corollary to Theorem 7.5. \square

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